Sum of two independent complex Wishart matrices with unequal covariances

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Abstract

In [1] we study the sum of two statistically independent complex Wishart matrices which separately comprise non-trivial external covariance matrices. They model the simplest set of time series with time dependent spatial correlations. We also consider a certain limit where one of the covariance matrices is degenerated. This case, called **half degeneracy**, follows a determinantal point process.
Our results are: a closed form for the k-point resolvent as a supermatrix integral & in the half degenerated case: an explicit expression for the correlation kernel.

1 Random matrix model

We introduce two copies of complex Gaussian matrices A and B:

- rectangular: $N \times N_A$ & $N \times N_B$, respectively,
- dimensions arbitrary, up to $N_A, N_B \geq N$,
- complex entries $A_{ij}, B_{ij} \in \mathbb{C}$.

The distribution for the entries:

$$\mathcal{P} = C_A C_B \, \mathrm{e}^{-\mathrm{Tr}\Sigma_A^{-1} AA^\dagger} \mathrm{e}^{-\mathrm{Tr}\Sigma_B^{-1} BB^\dagger}, \quad C_{A/B} \propto \det{}^{-N_{A/B}} \Sigma_{A/B} . \quad (1)$$

By construction, the **covariance matrices**,

$$\frac{1}{N_A}\left\langle AA^\dagger \right\rangle = \Sigma_A , \quad \frac{1}{N_B}\left\langle BB^\dagger \right\rangle = \Sigma_B , \quad (2)$$

are $N \times N$ positive definite. $\Sigma_{A/B}$ induce correlations into the entries of A and B, respectively. Interesting are the cross correlations, here the eigenvalues of

$$H = AA^\dagger + BB^\dagger . \quad (3)$$

2 Motivation

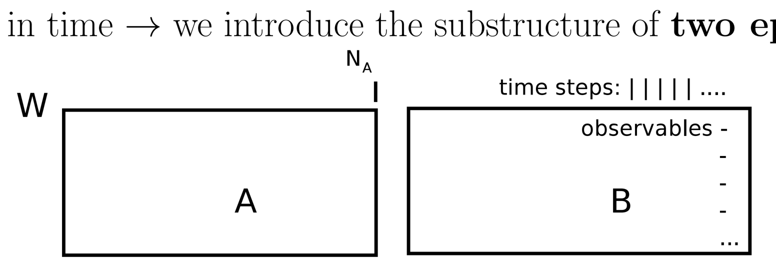
In a very general sense, time-series emerge by performing measurement in discrete time steps. Say we have N observables and N_W time steps, then after the measurement we obtain a rectangular data sample, we call it W. It is of the dimension: $N \times N_W$. In this simple set-up we ask for the correlations between the entries s,t and s',t'. For example we can take the estimator:

$$\langle W_{st} W_{s't'} \rangle_{\mathrm{data}} = \Sigma_{ss',tt'} . \quad (4)$$

The analysis of spatio-temporal correlations in W can be based on Σ, but we mostly need to simplify the situation for statistical models, for instance:

- Σ is trivial $\rightarrow$ no correlations.
- Σ factorises $\rightarrow$ time and spatial correlations are independent.
- $\Sigma_{ss',tt'} = \Sigma_{s,s}\delta_{tt'} \rightarrow$ no time dependence.

In our model: the time correlations are trivial, but the spatial correlations change in time $\rightarrow$ we introduce the substructure of **two epochs**.



The new epoch starts after the time step $t = N_A$ and different spatial correlations are assumed. They remain constant till $N_W = N_A + N_B$.

3 Starting point

They main results derived by S. Kumar in [2]:

- The distribution for the matrix H:

$$\mathcal{P}_H = C_H \det H^{N_W - N} \mathrm{e}^{-\mathrm{Tr}\Sigma_{A/B}^{-1} H} \times {}_1\mathcal{F}_1\left(N_{B/A}; N_W; \left(\Sigma_{A/B}^{-1} - \Sigma_{B/A}^{-1}\right) H\right) , \quad (5)$$

where ${}_1\mathcal{F}_1$ is the confluent hypergeometric function with matrix arguments.

- In the case of half degeneracy:

$$\Sigma_A = \sigma_A \mathbb{1}_N , \quad \Sigma_B = \mathrm{diag}\left(\sigma_{B1}, \sigma_{B2}, ..., \sigma_{BN}\right) , \quad (6)$$

the **joint probability distribution function** for the eigenvalues of H is given:

$$P_N(\lambda_1, ..., \lambda_N) = \frac{C_\Lambda}{\Delta_N(\delta_i)} \det_{N\times N}\left[\lambda_i^{j-1}\right] \det_{N\times N}\left[\varphi_j(\lambda_i)\right] \quad (7)$$

where we find the ordinary ${}_1F_1$ in the second determinant,

$$\varphi_j(\lambda) = \lambda^{N_W - N} \mathrm{e}^{-\sigma_A^{-1}\lambda} {}_1F_1\left(N_B + 1; N_W + 1; \delta_j \lambda\right) , \quad (8)$$

with the abbreviation $\delta_j = \sigma_A^{-1} - \sigma_{Bj}^{-1}$.

4 Determinantal point process

The joint probability distribution function (7) represents a determinantal point process. It is well known that the k-point correlation function,

$$R_k(\lambda_1, ..., \lambda_k) = \frac{N!}{(N-k)!} \prod_{l=k+1}^{N} \int_0^\infty d\lambda_l P_N(\lambda_1, ..., \lambda_N) , \quad (9)$$

can be written as a $k \times k$ determinant of the so-called **correlation kernel** $K_N(x,y)$:

$$R_k(\lambda_1, ..., \lambda_k) = \det_{k\times k}\left[K_N(\lambda_i, \lambda_j)\right] , \quad (10)$$

for example:

$k = N: \quad R_N(\lambda_1, ..., \lambda_N) = P_N(\lambda_1, ..., \lambda_N) \propto \det_{N\times N}\left[\lambda^{i-1}\varphi_j(\lambda)\right]$,

$k = 1: \quad R_1(x) = K_N(x,x)$ is the spectral density .

Thus, $K_N(x,y)$ is the desired quantity.

5 Method of Orthogonal polynomials

1. Construction of orthogonal polynomials:

$$K_N(x,y) = \sum_{l=1}^{N} p_{l-1}(x) q_l(y) , \quad \delta_{kl} = \int_0^\infty d\lambda p_{k-1}(\lambda) q_l(\lambda) .$$

2. Inversion of the Gram matrix:

$$K_N(x,y) = \sum_{l,j=1}^{N} x^{l-1} (g^{-1})_{lj} \varphi_j(y) , \quad g_{kl} = \int_0^\infty d\lambda \lambda^{k-1} \varphi_l(\lambda) .$$

3. Our approach:

$$K_N(x,y) = \sum_{j=1}^{N} p_{N-1}^{(j)}(x) \varphi_j(y) , \quad \delta_{kl} = \int_0^\infty d\lambda p_{N-1}^{(k)}(\lambda) \varphi_l(\lambda) . \quad (11)$$

All elements in the set $\left\{p_{N-1}^{(k)}\right\}_{k=1}^{N}$ are polynomials of the same degree, namely $N-1$. This set does not span the space of monomials in the first determinant of (7), as expected at first sight. The construction (11) in this form is new! Such an element $p_{N-1}^{(j)}$ is up to a constant the **average of a characteristic polynomial**:

$$p_{N-1}^{(j)}(x) \propto \left\langle \det\left[x\mathbb{1}_{N-1} - H'\right] \right\rangle^{(j)} , \quad (12)$$

where we denote with $(...)'$ a dimensional reduction: $N \rightarrow N-1$ and $N_B \rightarrow N_B - 1$. We exclude the element σ_{Bj} in (12).

6 Main idea

From the extended Andréief formula we know:

$$K_N(x,y) \propto \prod_{l=2}^{N} \int_0^\infty d\lambda_l \det_{N\times N}\begin{bmatrix} x^{j-1}\big|_{1\leq j\leq N} \\ \lambda_i^{j-1}\big|_{1\leq i,j\leq N}^{i\neq 1} \end{bmatrix} \det_{N\times N}\begin{bmatrix} \varphi_j(y)\big|_{1\leq j\leq N} \\ \varphi_j(\lambda_i)\big|_{1\leq i,j\leq N}^{i\neq 1} \end{bmatrix} .$$

From an algebraic decomposition of the determinants in the latter expression the correlation kernel (11) can be obtained:

- sum over j and the elements $\phi_j(y)$:

$$\det_{N\times N}\begin{bmatrix} \varphi_j(y)\big|_{1\leq j\leq N} \\ \varphi_j(\lambda_i)\big|_{1\leq i,j\leq N}^{i\neq 1} \end{bmatrix} = \sum_{j=1}^{N} (-1)^{j-1} \det_{(N-1)\times(N-1)}\left[\varphi_i(\lambda_k)\right] \varphi_j(y) ,$$

- characteristic polynomial as product of differences between x and the eigenvalues from the Vandermonde determinant:

$$\det_{N\times N}\begin{bmatrix} x^{j-1}\big|_{1\leq j\leq N} \\ \lambda_i^{j-1}\big|_{1\leq i,j\leq N}^{i\neq 1} \end{bmatrix} = \det_{(N-1)\times(N-1)}\left[\lambda_i^{j-1}\right] \prod_{k=2}^{N} (\lambda_k - x) ,$$

- the identification of the remaining determinants to the joint probability distribution function of $N-1$ eigenvalues, eq. (7):

$$\det_{(N-1)\times(N-1)}\left[\lambda_i^{j-1}\right] \det_{(N-1)\times(N-1)}\left[\varphi_i(\lambda_k)\right] \propto P_{N-1}^{(j)}(\lambda_1, ..., \lambda_N) .$$

7 Supersymmetry method

The supersymmetry method can be applied for the calculation of the average of the characteristic polynomial. In particular, we use the superbosonisation technique and obtain:

$$\left\langle \det\left[x\mathbb{1}_N - H\right] \right\rangle = N_A! N_B! \oint_{\gamma_1} \frac{dz_1}{2\pi i} \oint_{\gamma_2} \frac{dz_2}{2\pi i} \frac{\mathrm{e}^{z_1 + z_2}}{z_1^{N_A+1} z_2^{N_B+1}} \times \prod_{k=1}^{N} \left(x - z_1\sigma_A - z_2\sigma_{Bk}\right) . \quad (13)$$

Moreover, superbosonisation is much more stronger: the ratio of arbitrary many characteristic polynomials,

$$Z_{q|p}\left(x_1, ..., x_p; y_1, ..., y_q\right) := \left\langle \frac{\prod_{j=1}^{p} \det\left[x_j - H\right]}{\prod_{j=1}^{q} \det\left[y_j - H\right]} \right\rangle , \quad (14)$$

with arbitrary covariances can be obtained:

$$Z_{q|p} = C_{N_A} C_{N_B} \int d\mu(U_A) \int d\mu(U_B) e^{-\mathrm{Str}U_A - \mathrm{Str}U_B} \mathrm{Sdet}^{N_A} U_A \mathrm{Sdet}^{N_B} U_B \times \mathrm{Sdet}^{-1}\left[\mathbb{1}_N \otimes \begin{bmatrix} \mathrm{Y} & 0 \\ 0 & \mathrm{X} \end{bmatrix} - \Sigma_A \otimes U_A - \Sigma_B \otimes U_B\right] . \quad (15)$$

The resulting supermatrix model $Z_{q|p}$ is the **generating function** for the k-point resolvent. Especially the Green's function is available with $G(x) = \partial_x Z_{1|1}(x,y)\Big|_{y=x}$.

8 Results

- The spectral density is given by $R_1(x) = K_N(x,x)$.

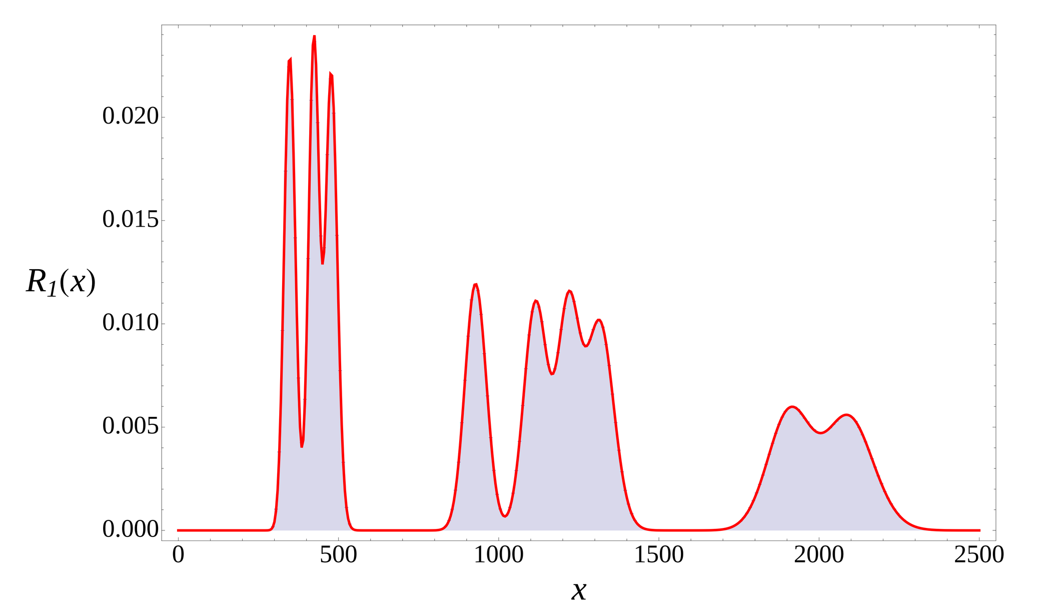


Figure 2:
○ *Dimensions of the time series:* $N = 9$, $N_A = 350$ *&* $N_B = 400$.
○ *The covariance values are set to:* $\sigma_A = 1$ *and* $\Sigma_B = \mathrm{diag}\left(0.02, 0.2, 0.3, 1.5, 2.01, 2.25, 2.27, 4.05, 4.03\right)$.
○ *The red curve is the analytic result, blue dashed area is a histogram from* 10^6 *"Monte Carlo matrices"*

- We have an explicit form for the correlation kernel K_N and thus all k-point correlation functions are accessible.
- The generating function $Z_{q|p}$ yields all possible k-point resolvents.
- The method of orthogonal polynomials shown in section 6 is applicable for every polynomial ensemble with the symmetry $\varphi_j(\lambda) = f\left(\delta_j \lambda\right)$.

9 Open problems

- Arbitrary many epochs: $H = \sum_{i=1}^{T} A_i A_i^\dagger$?
- Singular values of temporal cross-correlations $A^\dagger B$?
- micr. properties, doubly correlated Wisharts, $\beta = 1$ or 4, ...?

10 Refererences

[1] G. Akemann, T. Checinski & M. Kieburg, submitted to J. Phys. A: Math. & Theor. (2015) [arXiv:1509.03446 [math-ph]]

[2] S. Kumar, EPL **107**, 60002 (2014) [arXiv:1406.6638 [math-ph]]