Truncations of random unitary matrices revisited

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joint review with H.-J. Sommers:
Non-Hermitian Random Matrix Ensembles, arXiv:0911.5645,
to appear in the Oxford Handbook of Random Matrix Theory
Choose a unitary matrix at random and partition it:

\[ U = \begin{pmatrix} T & S \\ Q & R \end{pmatrix} \mapsto T \quad T \text{ is } m \times p \]

‘Choose a unitary matrix at random’ - implicit referral to uniform distribution on the group surface, known as the Haar measure.

Thus, consider the unitary group \( U(n) \) equipped with the normalised Haar measure \( d\mu_H(U) \). The property of invariance determines this measure uniquely. This can be used for sampling of the Haar distribution via Gram-Schmidt.

The Haar measure induces a probability distribution \( d\rho_{n,m \times p}(T) \) on truncated unitaries.
What truncations are good for?

(1) **Quantum transport problems** (Beenakker’97, poster by Nick Simm)

Additive stats of EVs of $TT^\dagger$ describe phys quantities of interest, i.e. $\text{tr} TT^\dagger$ for conductance of quasi one-dimensional wires.

(2) **Open chaotic sys** (Fyodorov & Sommers, ’97 Życzkowski & S. ’00)

Eigenvalues of $T$ are used to model resonances.

(3) **Combinatorics of vicious walkers** (Novak ’09)

$< |\text{tr} T|^N>_T$ enumerates configs of random-turn turn vicious walkers.

(4) **Random determinants** (Fyodorov & K., ’07), e.g.,

$$< \frac{1}{|\text{det}(I-zA)|^{2m}}>_A = \int < \frac{1}{\text{det}(I-|z|^{2TT^\dagger} \otimes AA^\dagger)}>_A \text{d}\rho_{n,m\times m}(T)$$

for complex random $n \times n$ matrices $A$ with invariant distribution.

Singular values of $T$ (1,4); eigenvalues of $T$ (2,3)
Truncation map: \( U \mapsto T \), \( U \) is \( n \times n \), \( T \) is \( m \times p \), \( m \leq p \)

Have \( TT^\dagger + SS^\dagger = I \) by unitarity. If \( n \geq m + p \) then (generically) \( SS^\dagger \) has rank \( m \) and the image of \( U(n) \) is the entire matrix ball \( TT^\dagger \leq I \).

**Theorem 1** *(Friedman&Mello ’85, Fyodorov&Sommers ’03, Forrester ’06)*

For \( n \geq m + p \)

\[
d\rho_{n,m \times p}(T) \propto \det(I - TT^\dagger)^{n-m-p} \chi_{TT^\dagger \leq I}(T) dT
\]

where \( dT \) is the Cartesian volume element in \( \mathbb{C}^{m \times p} \). For invariant \( f \)

\[
\int_{\mathbb{C}^{m \times p}} f(TT^\dagger) d\rho_{n,m \times p}(T) = \text{const.} \times
\]

\[
= \int_{\mathbb{C}^{m \times m}} f(ZZ^\dagger) \det(ZZ^\dagger)^{p-m} \det(I - ZZ^\dagger)^{n-m-p} \chi_{ZZ^\dagger \leq I}(Z) dZ
\]
If $n < m + p$ (e.g., deleting just a few columns/rows) then $\lambda = 1$ is an eigv of $TT^\dagger$ of multiplicity $m + p - n$. Hence the image of $U(n)$ is a set on the boundary of $TT^\dagger \leq I$.

Useful explicit expression for $d\rho_{n,m \times p}(T)$ is unknown. However:

**Theorem 2** *(Fyodorov & K. ’07)* Let $n < m + p$. Then for invariant $f$

$$
\int f(TT^\dagger)d\rho_{n,m \times p}(T) = \text{const.} \times
$$

$$
\int f \left( \begin{pmatrix} ZZ^\dagger & 0 \\ 0 & I \end{pmatrix} \right) \det(ZZ^\dagger)^{p-m} \det(I - ZZ^\dagger)^{m+p-n} \chi_{ZZ^\dagger \leq I(Z)} dZ
$$

*matrices $T$ are $m \times p$ and $Z$ are $(n - p) \times (n - p)$*

SVs of truncations: with Thms 1 and 2 in hand one can study distr. of (nontrivial) eigenvalues of $TT^\dagger$. 
Consider square truncations $T (m \times m)$, these are random contractions.

**Theorem 3** (Życzkowski & Sommers ’00) The symmetrised jpdf of the EVs of $T$ is

$$P(z_1, \ldots, z_m) \propto \prod_{j=1}^m w(z_j) \prod_{1 \leq j < k \leq m} |z_j - z_k|^2$$

with weight function $w(z) = (1 - |z|^2)^{n-m-1} \chi_{|z|<1}(z)$

Ż & S didn’t use $d\rho_{n,m \times m}$. Equally, (1) can be obtained via Thm 1,2, e.g. Forrester & Krishnapur ’09 for $n > 2m$.

Two interesting regimes: (i) $n \to \infty$, $n - m = O(1)$ (weak non-unitarity) and (ii) $n \to \infty$, $m/(n - m) = \alpha = O(1)$ (strong non-unitarity)
Eigenvalue correlation functions

These are just marginals of the jpdf:

$$R_k(z_1, \ldots, z_k) = \frac{m!}{(m-k)!} \int d^2 z_{k+1} \ldots \int d^2 z_m P(z_1, \ldots, z_k, z_{k+1}, \ldots, z_m),$$

The EV corr fncts for truncations can be obtained by the method of OPs.

For the rotation invariant weights, $w(z) = w(|z|)$, OPs are just powers $z^l$:

$$\int d^2 z \ w(z) \ z^i z^* j = h_j \delta_{i,j},$$

leading to

$$R_k(z_1, \ldots, z_k) = \prod_{l=1}^{m} w(z_l) \ \text{det}(K(z_i, z_j)); \quad K(u,v) = \sum_{l=0}^{m-1} \frac{(uv^*)^l}{h_l}$$

For truncated unitaries the sum on the rhs is the binomial series for $(1 - uv^*)^{-(n-m+1)}$ truncated after $m$ terms. This gives the kernel in terms of the incomplete Beta function.
Incomplete Beta fnc:

\[ I_x(a, b) = \frac{1}{B(a, b)} \int_0^x t^{a-1}(1 - t)^{b-1} \, dt \]

We have

\[ K(u, v) = \frac{n - m}{\pi} \frac{I_{1-uv^*}(n - m + 1, m)}{(1 - uv^*)^{n-m+1}} \]

This representation seems to be new. It is convenient for asymptotic analysis. Also one can handle more general \( w(z) = |z|^{2p}(1 - |z|^2)^q \).

Compare with the complex Ginibre \((d\mu(J) \propto e^{-\text{tr} J J^\dagger} dJ)\). There \( w(z) = e^{-|z|^2} \), have truncated exponential series for the kernel:

\[ K(u, v) = \frac{n}{\pi} e^{uv^*} \frac{\Gamma(n, uv^*)}{\Gamma(n)} \text{ with } \Gamma(n, x) = \int_x^\infty e^{-t} t^{n-1} dt \]
Consider $n, m \to \infty$, $m/(n - m) = \alpha > 0$.

The EVs of truncated unitaries are distributed inside the disk $|z|^2 \leq \frac{\alpha}{(1+\alpha)}$
with density (Życzkowski, Sommers, '00)

$$R_1(z) \simeq \frac{m}{\pi \alpha} \frac{1}{(1 - |z|^2)^2}.$$
Strong non-unitarity: boundary of EV distribution

When one traverses the boundary of the support of the limiting distribution the EV density vanishes at a Gaussian rate. In the transitional region

\[ R_1 \left( \sqrt{\frac{\alpha}{1 + \alpha}} + \frac{x}{\sqrt{m}} \right) \approx \frac{m}{2\pi} \frac{(1 + \alpha)^2}{\alpha} \text{erfc} \left( \sqrt{2} \frac{1 + \alpha}{\sqrt{\alpha}} x \right) \]

Same Erfc Law as in Ginibre (also poster by Navinder Singh). The average no. of EVs outside the support \( \approx \sqrt{m(1 + \alpha)/(2\pi)} \).
Strong non-unitarity, locally at the origin

Scale $z$ with the mean dist between EVs, $z_j = \frac{u_j}{\sqrt{R_1(0)}}, |u_j| = O(1)$.

Rescaled corr fncs are given by Ginibre's: $e^{-\pi \sum_j u_j^* u_j^2 \det(e^{\pi u_i u_j^*})}$.

**Density of nearest-neighbour distances:** $p(s) = -\frac{d}{ds} H(s)$ where

$$H(s) = \frac{m}{R_1(0)} \int d^2 z_2 \cdots \int d^2 z_m P(0, z_2, \ldots, z_m) \prod_{j=2}^{m} (1 - \chi_D(z_j)).$$

(cond. prob. given one EV at $z = 0$ all others outside the disk $D \ |z| < s$)

$H(s)$ can be found by making use of rotational invariance

$$H(s) = \prod_{j=1}^{m-1} (I_{1-s^2(n-m, j+1)}) \simeq \prod_{j=1}^{\infty} \frac{\Gamma(j + 1, \pi x^2)}{\Gamma(j + 1)}, \quad s = \frac{x}{\sqrt{R_1(0)}},$$

Have **cubic law** of EV repulsion as in Ginibre
(Grobe, Haake & Sommers'88)
Assume \( n = m + l, m \to \infty, \) \( l \) is finite (delete a few rows & columns)

In this limit the EVs of truncated Haar unitaries lie close to the unit circle with the typical distance being \( O(1/m) \).
Scaling \( z \) accordingly, \( z_j = \left(1 - \frac{y_j}{m}\right) e^{i\varphi_0 + i\frac{\varphi_j}{m}} \), one finds the EV density

\[
R_1(z) \simeq \frac{m^2}{\pi} \frac{(2y)^{l-1}}{(l-1)!} \int_0^1 e^{-2yt} t^l \, dt, \quad m \to \infty \text{ and } l \text{ is finite.}
\]

(Życzkowski & Sommers, ’00) and correlations

\[
R_k(z_1, \ldots, z_k) \simeq \left(\frac{m^2}{\pi}\right)^k \prod_{j=1}^k \frac{(2y_j)^{l-1}}{(l-1)!} \det \left( \int_0^1 e^{-(y_i+y_j+i(\varphi_i-\varphi_j))t} t^l \, dt \right)
\]

This is a particular case of a ’universal’ expression describing EV correlations for random contractions (Fyodorov & Sommers ’03).

Interestingly, a different ensemble, \( J = H + i\gamma W \), leads to the same form of correlations (Fyodorov & K. ’99). Here \( H \) is drawn from the GUE, \( \gamma > 0 \) and \( W \) is a diagonal matrix with \( l \) 1’s and \( m \) zeros.
A simple model leading to universal eigenvalue statistics.