# RMT benchmark for spectra of delayed correlation matrices

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## Abstract

We study the complex spectrum and the certain eigenvector correlator of the time-lagged correlation matrices, corresponding to the absence of any structure in the data. To this end we solve in the large $N$ limit a more general problem - the spectrum of $\frac{1}{T}XAX^\dagger$, with no constrains on $A$.

## Correlation matrices

In the analysis of time series one often studies the cross-correlation matrices. They measure correlation between variables at the same moment of time.

$$C_{ij} = \frac{1}{T}\sum_{t=1}^{T} x_{it}x_{jt} = \frac{1}{T}XX^\dagger.$$

Null hypothesis (absence of correlations) - Wishart ensemble.
In order to find correlation between variables at different moments of time, one has to study the time-lagged correlation matrix

$$C_{ij}^{\tau} = \frac{1}{T-\tau}\sum_{t=1}^{T-\tau} x_{i,t}x_{j,t+\tau} = \frac{1}{T-\tau}(XDX^\dagger)_{ij},$$

with $D_{ts} = \delta_{s,t+\tau}$ - delay matrix. Null hypothesis - $X$ a rectangular Gaussian random matrix. We are interested in the limit $N, T, \tau \to \infty$ with $r = N/T$ *and* $\tau/T$ fixed (double scaling).

## Previous approaches

- Symmetrization of the lagged correlation matrix $C^{sym} = \frac{1}{2}(C^\tau + C^{\tau\dagger})$. (Mayya, Amritkar, Krakow group 2006)
- Whitening. Two time series $x_{it}$ and $y_{it} = y_{i,t+\tau}$, removal of the equal-time correlations by decomposing cross-correlation matrix $C_x = \frac{1}{T}xx^\dagger = U\Lambda U^\dagger$ and defining new time series $x' = \Lambda^{-1/2}U^\dagger x$, analogously for $y$. Then the equal-time correlations are trivial ($\frac{1}{T}x'x'^\dagger = 1$). In the absence of lagged correlations the product $\frac{1}{T-\tau}xy^\dagger$ reduces to the product of free Jacobi. (Bouchaud 2007)
- Abelization. Solution to the non-Hermitian problem from the solution of the symmetrized problem with the assumption of the azimuthal symmetry of the complex spectrum. (Biely, Thurner 2006) Additional assumption needed - matrices are normal.
- Diagrammatic calculations for a vast number of models (Jarosz 2010, posted only on arXiv).
- Product of two independent rectangular random matrices (Livan 2012).

## Setting the stage: non-Hermitian random matrices [1]

The main object of interest is the mean spectral density

$$\rho(z,\bar{z}) = \left\langle \frac{1}{N}\sum_{i=1}^{N}\delta^{(2)}(z-\lambda_i)\right\rangle,$$

encoded in the object called the quaternionic Green's function

$$G(Q) = \begin{pmatrix} g & i\bar{v} \\ iv & \bar{g}\end{pmatrix} = \left\langle \frac{1}{N}\mathrm{bTr}\begin{pmatrix} z-X & i\bar{w} \\ iw & \bar{z}-X^\dagger\end{pmatrix}^{-1}\right\rangle, \qquad (1)$$

where

$$\mathrm{bTr}\begin{pmatrix} A & B \\ C & D\end{pmatrix} = \begin{pmatrix} \mathrm{tr}A & \mathrm{tr}B \\ \mathrm{tr}C & \mathrm{tr}D\end{pmatrix}.$$

One recovers the spectral density via

$$\rho(z,\bar{z}) = \lim_{w\to 0}\frac{1}{\pi}\partial_{\bar{z}}g(z,\bar{z},w,\bar{w}).$$

Moreover, in the large $N$ limit

$$\frac{1}{\pi}\lim_{w\to 0}|v|^2 = O(z,\bar{z}) = \lim_{N\to\infty}\left\langle\frac{1}{N^2}\sum_{i=1}^{N}\langle L_i|L_i\rangle\langle R_i|R_i\rangle\,\delta^{(2)}(z-\lambda_i)\right\rangle$$

gives the correlator of left ($\langle L_i|$) and right ($|R_i\rangle$) eigenvectors, playing the role in the stability of the spectrum.

## References

[1] R. A. Janik, M. A. Nowak, G. Papp, I. Zahed, Nucl. Phys. B 501, 603642 (1997).
[2] M. A. Nowak, W. Tarnowski *Spectra of large time-lagged correlation matrices from random matrix theory*. Shortly on arXiv.

## Solution to the problem [2]

$2\times 2$ algebraic matrix equation for the quaternionic Green's function $G(Q)$

$$\left[Q - \frac{\alpha}{T}\mathrm{bTr}\left(\mathcal{D}\left[\mathbb{1}_{2T} - \alpha r(G\otimes\mathbb{1}_T)\mathcal{D}\right]^{-1}\right)\right]G = \mathbb{1}_2.$$

The LHS is understood in the limit $N, T, \tau \to \infty$ with $r = N/T$ and $\tau/T$ fixed. Here

$$Q = \begin{pmatrix} z & i\bar{w} \\ iw & \bar{z}\end{pmatrix}, \quad \mathcal{D} = \begin{pmatrix} D & 0 \\ 0 & D^\dagger\end{pmatrix}, \quad \alpha = \frac{T}{T-\tau}.$$

If $D$ is invertible

$$[Q + \alpha G_{A^{-1}}(\alpha r G)]\,G = \mathbb{1}_2. \qquad (2)$$

Derivation: Wick's theorem + combinatorics behind Feynman diagrams.
The spectrum of the lagged correlation matrix always possesses the rotational symmetry.

## Unit time shift

A particular example where the solution can be applied corresponds to $\tau = 1$. In this case we obtain the cubic equation for the radial cumulative distribution function $f(s) = 2\pi\int_0^s \rho(r)rdr$:

$$4f^3r^3 + 4f^2r^2(1-r) + fr\left((1-r)^2 - |z|^2\right) - |z|^2 = 0$$

Having solved this equation, one calculates the spectral density via $\rho(|z|) = \frac{1}{2\pi|z|}f'(|z|)$. The spectral radius is equal to $\sqrt{r(r+1)}$.

## Spectral radius

Given by the solution of the algebraic equation

$$\sum_{k=1}^{M-1}\left(\frac{\alpha r}{s_{ext}}\right)^{2k}(1-k\beta) = r,$$

where $\beta = \tau/T$, $M = \lceil\frac{T}{\tau}\rceil$ and $\lceil x\rceil$ is the ceiling function.

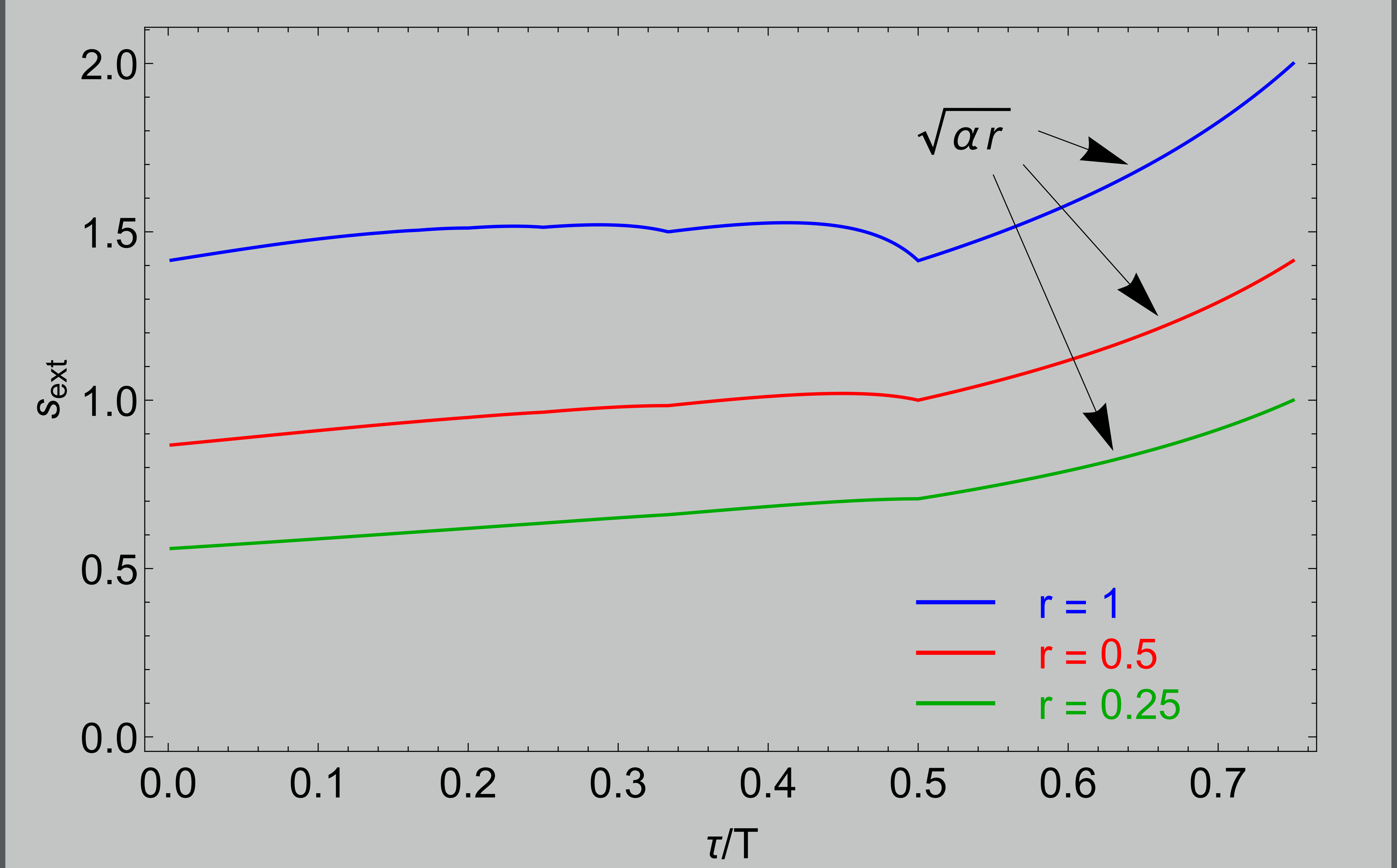


Figure 1: Radius of the support of the spectral density as a function of the relative lag depth for several rectangularities.

## Conclusions

- We derived the algebraic equation for the quaternionic Green's function in the limit $N, T, \tau \to \infty$ in the double scaling $r = \frac{N}{T}$ and $\beta = \frac{\tau}{T}$ fixed.
- The result is valid for an arbitrary, not necessarily symmetric matrix, sandwiched between two rectangular Gaussian matrices.

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