The Limit of Weak non-Hermiticity Revisited: Beyond the Elliptic Ensemble

Martin Venker

UC Louvain

(joint work with G. Akemann and M. Cikovic)

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Hermitian vs. Non-Hermitian RMT

- “Hermitian RMT”: one-dimensional eigenvalues, “non-Hermitian RMT”: two-dimensional eigenvalues.
- Dimension of the entries matters! Matrices with complex entries have different statistics than matrices with real entries.
- No spectral calculus! How to define and deal with matrix models?

In this talk: Random matrices with complex entries and without symmetries.

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The Elliptic Ensemble

- $J := \sqrt{1+\tau} J_1 + i\sqrt{1-\tau} J_2$, $J_{1,2}$ independent GUE matrices, $\tau \in (-1,1)$. Special case $\tau = 0$: Ginibre Ensemble.
- Its density is

$$P_{N,\mathrm{ell}}(J) := \frac{1}{Z_{N,\mathrm{ell}}} \exp\left[-\frac{N}{1-\tau^2} \mathrm{Tr}\left(JJ^* - \frac{\tau}{2}(J^2 + J^{*2})\right)\right].$$

- Interpolates between Ginibre ($\tau = 0$) and GUE ($\tau = 1$).

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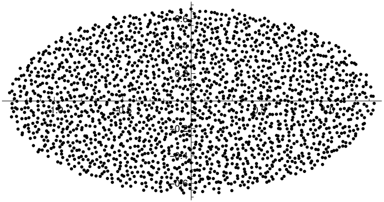

The elliptic law

(one realization for $N = 2000$ and $\tau = 1/3$)

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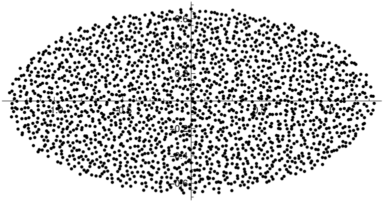

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- Matrices with independent entries easy to define.
- What about models with exploitable (eigenvalue) density?
- First attempt: Density on $\mathbb{C}^{N\times N}$ prop. to $\exp[-N\mathrm{Tr}\,(JJ^*)^k]$?

  Problem: $\mathrm{Tr}\,JJ^*$ depends not only on eigenvalues, but also on (for $k>1$ dependent) additional variables (from Schur decompos.).
- Additional variables decouple for models

$$\propto \exp\left[-c_1 N\mathrm{Tr}\,(JJ^*) + c_2 N\,\Re(\mathrm{Tr}\,V(J))\right],$$

  V potential. **Problem**: Normalizable (almost) only for $V(J)=J^2$.

  (Elbau,Felder '05; Bleher,Kuijlaars '12)
- Consider normal random matrices: Spectral calculus $\rightarrow$ determinantality $\rightarrow$ universality (Ameur,Hedenmalm,Makarov '11)

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Our Model

Let $\gamma \geq 0, K_p \in \mathbb{R}$ fixed, $\tau \in [0,1)$ (or $(-1,1)$) and define

$$P_{N,\mathrm{Tr}^2}(J) := \frac{1}{Z_{N,\mathrm{Tr}^2}} \exp\left[-\frac{N}{1-\tau^2}\mathrm{Tr}\left(JJ^* - \frac{\tau}{2}(J^2 + J^{*2})\right) - \gamma(\mathrm{Tr}\, JJ^* - N\,K_p)^2\right].$$

Comments:

- Model penalizes/punishes deviations of $\mathrm{Tr}\, JJ^*/N$ from the value K_p.
- $\gamma = 0$: Elliptic ensemble. $\gamma \to \infty$: a fixed trace ensemble on $\{J : \mathrm{Tr}\, JJ^* = K_p\}$ (future work).
- Model is not Gaussian.
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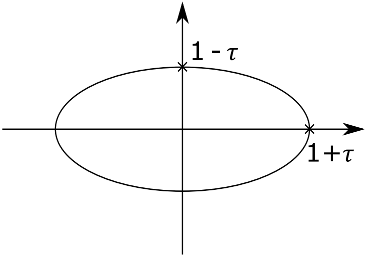
Elliptic ensemble: As $N \to \infty$, the eigenvalues spread uniformly in



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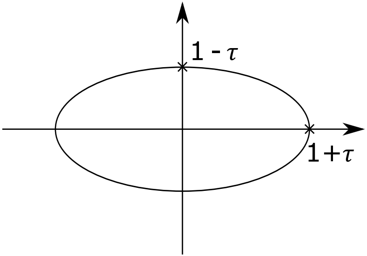

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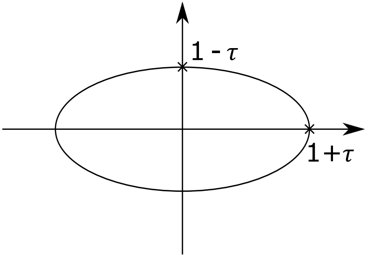

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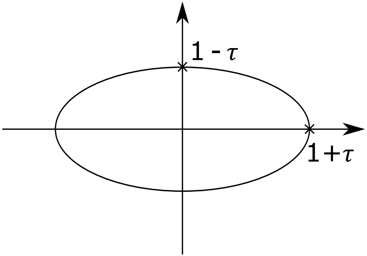

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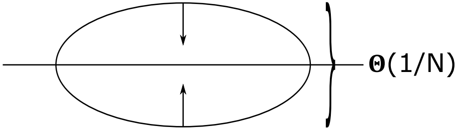

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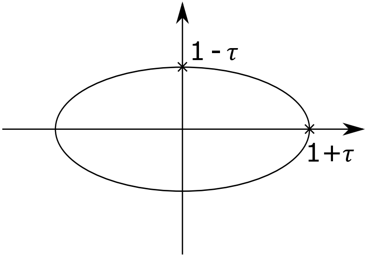

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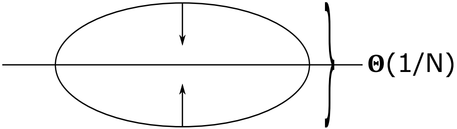

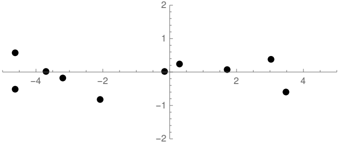

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In local scale, eigenvalues live in $\mathbb{C}$.

Strong non-Hermiticity: Global Result

Let ρ^k_{N,Tr^2} be the k-th correlation function of the eigenvalue distribution of P_{N,Tr^2}: If $P_{N,\mathrm{Tr}^2,\mathrm{EV}}$ is eigenvalue density on $\mathbb{C}^N$,

$$\rho^k_{N,\mathrm{Tr}^2}(z_1,\dots,z_k) := \frac{N!}{(N-k)!}\int_{\mathbb{C}^{N-k}} P_{N,\mathrm{Tr}^2,\mathrm{EV}}(z)dz_{k+1}\dots dz_N.$$

Theorem (Akemann,Cikovic,V. '16)

Let $\tau \in [0,1)$ be fixed. Then there are constants $c_1, c_2, C > 0$, depending on K_p, γ and τ such that with $E := \{Z : c_1(\Re Z)^2 + c_2(\Im Z)^2 \leq 1\}$ the following holds: For any $Z \in \mathbb{C}$, $Z \notin \partial E$, we have

$$\lim_{N\to\infty}\frac{1}{N}\rho^1_{N,\mathrm{Tr}^2}(Z) = \mathbb{1}_{E^\circ}(Z)\cdot\frac{C}{\pi}.$$

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Theorem (ACV '16)

Same C, same elliptic set E: For $k = 1, 2 \ldots, Z \in E^\circ$, $z_1, \ldots, z_k \in \mathbb{C}$, as $N \to \infty$

$$\frac{1}{(CN)^k} \rho^k_{N,\mathrm{Tr}^2} \left(Z + \frac{z_1}{\sqrt{CN}}, \ldots, Z + \frac{z_k}{\sqrt{CN}} \right) = \det \left(K_{\mathrm{strong}}(z_j, z_l) \right)_{j,l \le k} + \mathcal{O}\left(\frac{1}{\sqrt{N}} \right), \quad \textit{where}$$

$$K_{\mathrm{strong}}(z_j, z_l) := \frac{1}{\pi} \exp \left[- \frac{|z_j|^2 + |z_l|^2}{2} + z_j \overline{z_l} \right].$$

The $\mathcal{O}$ term is uniform for Z from any compact subset of E° and $z_1, \ldots, z_k$ from compacts of $\mathbb{C}$.

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- Same limit shown for other ensembles:
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$$\lim_{N\to\infty} \frac{1}{N}\rho^1_{N,\mathrm{Tr}^2}(Z) = 0,$$

and for any $X \in \mathbb{R}$

$$\lim_{N\to\infty} \int_{\mathbb{R}} \frac{1}{N}\rho^1_{N,\mathrm{Tr}^2}(X+iY)dY = \frac{C}{2\pi}\sqrt{\frac{4}{C} - X^2}\,\mathbb{1}_{[-\frac{2}{\sqrt{C}},\frac{2}{\sqrt{C}}]}(X).$$

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$$\frac{1}{(N\nu(X))^{2k}}\rho^k_{N,\mathrm{Tr}^2}\left(X + \frac{z_1}{N\nu(X)}, \ldots, X + \frac{z_k}{N\nu(X)}\right)$$

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The $\mathcal{O}$ *term is uniform for* $X \in (-\frac{2}{\sqrt{C}} + \delta, \frac{2}{\sqrt{C}} - \delta)$ *for any* $\delta > 0$ *fixed and any* $z_j, j = 1, \dots, k$ *chosen from an arbitrary compact subset of* $\mathbb{C}$.

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- Transition to sine kernel statistics ($\alpha \to 0$):

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$$P_{N,\mathrm{Tr}^2}(J) =$$

$$\frac{\exp\left[-N\left(\frac{1}{1-\tau^2} \qquad \right)\mathrm{Tr}\, JJ^* + \frac{\tau N}{2(1-\tau^2)}\mathrm{Tr}\left(J^2 + {J^*}^2\right) - \gamma(\mathrm{Tr}\, JJ^* - N(K_p \qquad))^2\right]}{Z_{N,\mathrm{Tr}^2}}$$

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Choose K such that $\mathrm{Tr}\,JJ^* - N(K_p+K)$ is small under $P_{a,b}$.

Proof: Recentering II

Note

$$\frac{Z_{N,\mathrm{Tr}^2} \exp\left[-\gamma N^2(K^2 + 2KK_p)\right]}{Z_{a,b}} = \mathbb{E}_{a,b} \exp\left[-\gamma(\mathrm{Tr}\,\tilde{J}\tilde{J}^* - N(K_p + K))^2\right],$$

$\tilde{J}$ integration variable.

Lemma

There is a unique K s.t. for some constants C_1, C_2

$$0 < C_1 \leq \mathbb{E}_{a,b} \exp\left[-\gamma(\mathrm{Tr}\,\tilde{J}\tilde{J}^* - N(K_p + K))^2\right] \leq C_2 < \infty.$$

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Now use

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where $a(t) := a - i\sqrt{\gamma}t$.

Proof: Linearization II

Let $\rho^k_{a(t),b}$ be the k-th correlation function of the eigenvalue density of $P_{a(t),b}$ (and ρ^k_{N,Tr^2} that of P_{N,Tr^2}). The linearization gives

$$\rho^k_{N,\mathrm{Tr}^2}(z) - \det(K_{\mathrm{w/s}}(z_j, z_l))_{j,l\leq k} =$$
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Advantage: $\rho^k_{a(t),b}$ is determinantal, i.e.

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