A linear statistics with infinite moments – Wigner time delay distribution in multichannel disordered wires

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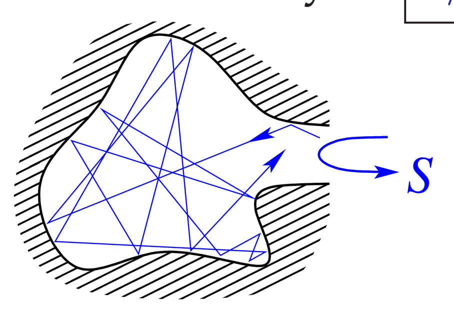
Scattering in Chaotic QDs *versus* Disordered wires

Wigner-Smith time-delay matrix :

Scattering matrix $\mathcal{S}(\varepsilon)$ $\longrightarrow$ $\mathcal{Q} \stackrel{\text{def}}{=} -\mathrm{i}\hbar\mathcal{S}^\dagger \dfrac{\partial\mathcal{S}}{\partial\varepsilon}$

Random Matrix Theory $\Rightarrow$ $1/\mathcal{Q} \in$ Laguerre ensemble

Chaotic quantum dots:

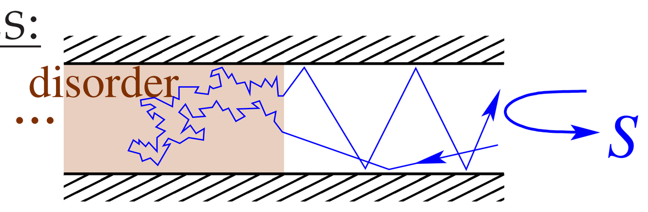


$$\mathcal{P}_N(\gamma_1,\cdots,\gamma_N) \propto \prod_{i<j}|\gamma_i-\gamma_j|^\beta \prod_{k=1}^N \gamma_k^{\beta N/2}\, e^{-\beta\gamma_k/2}$$

Eigenvalue of $\mathcal{Q}$: proper time $\tau_i = \tau_{\mathrm{H}}/\gamma_i$ ($\tau_{\mathrm{H}} = \frac{h}{\Delta}$)

Brouwer, Frahm & Beenakker, PRL (1997)

Disordered wires:



$$\mathcal{P}_N(\gamma_1,\cdots,\gamma_N) \propto \prod_{i<j}|\gamma_i-\gamma_j|^\beta \prod_{k=1}^N e^{-\beta\gamma_k/2}$$

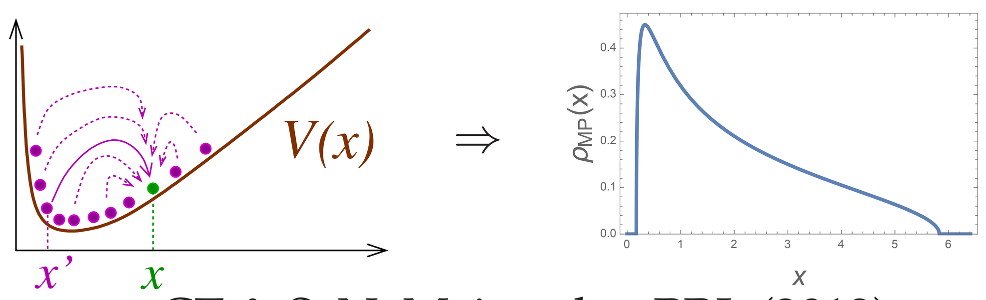
Beenakker & Brouwer, Physica E (2001)

Coulomb gas method

Density of eigenvalues $\rho(x) = \frac{1}{N}\sum_i \delta(x-\frac{\gamma_i}{N})$ has weight $\mathcal{P}_N(\gamma_1,\cdots,\gamma_N) \propto \exp\left\{-\frac{\beta}{2}N^2\mathscr{E}[\rho(x)]\right\}$ where

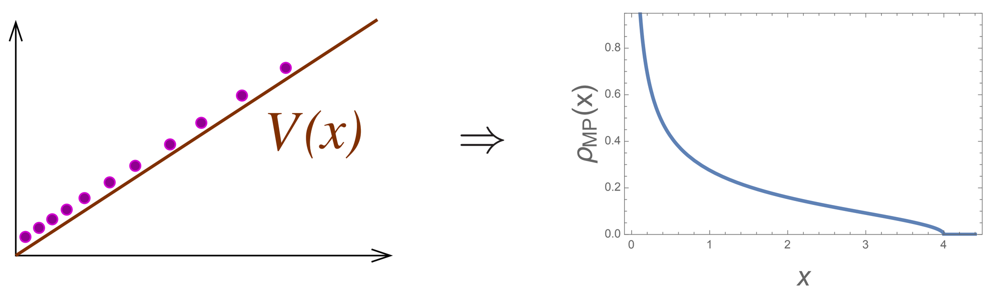
$$\mathscr{E}[\rho(x)] = \int_0^\infty \mathrm{d}x\,\rho(x)\,\underbrace{(x-\theta\ln x)}_{\stackrel{\text{def}}{=}V(x)} - \int_0^\infty \mathrm{d}x\mathrm{d}x'\,\rho(x)\,\rho(x')\,\ln|x-x'| \qquad (1)$$

Chaotic quantum dots ($\theta=1$):



CT & S. N. Majumdar, PRL (2013)

Disordered wires ($\theta=0$):



A. Grabsch & CT, J.Phys.A (2016)

Linear statistics

Linear statistics :

$$s = \frac{1}{N}\sum_{i=1}^N f(x_i) \longrightarrow \tau_{\mathrm{W}} = \frac{\mathrm{Tr}\{\mathcal{Q}\}}{N} = \frac{\tau_{\mathrm{H}}}{N}\sum_i \frac{1}{\gamma_i}$$

Distribution and large deviation function :

$$P_N^{(\beta)}(s) = \frac{\int\mathcal{D}\rho\, e^{-\frac{\beta}{2}N^2\mathscr{E}[\rho]}\,\delta(\int\rho-1)\,\delta(\int\rho f - s)}{\int\mathcal{D}\rho\, e^{-\frac{\beta}{2}N^2\mathscr{E}[\rho]}\,\delta(\int\rho-1)} \underset{N\to\infty}{\sim} \exp\left\{-\frac{\beta N^2}{2}\Phi(s)\right\} \qquad (2)$$

where the **large deviation function** is

$$\Phi(s) = \mathscr{E}[\rho^\star(x;s)] - \mathscr{E}[\rho_{\mathrm{MP}}(x)] \qquad (3)$$

Typical value s_* :

$$\Phi'(s_*) = 0 \qquad (\text{i.e. } \rho^\star(x;s_*) = \rho_{\mathrm{MP}}(x)) \qquad (4)$$

The problem: Infinite moments

For disordered wires ($\theta=0$):

$$s_* = \int\frac{\mathrm{d}x}{x}\,\rho_{\mathrm{MP}}(x) = \infty \qquad (5)$$

$\leftrightarrow$ $\Phi(s)$ is *monotonous*... $P_N^{(\beta)}(s) = ?$

Question :
What is the distribution of $s = \sum_i \gamma_i^{-1}$?

Beyond the large deviations : a conjecture

- Equilibrium condition :

$$\frac{\delta}{\delta\rho(x)}\left[\mathscr{E}[\rho] + \mu_0\left(\int\mathrm{d}x\,\rho(x)-1\right) + \mu_1\left(\int\mathrm{d}x\,\rho(x)\,f(x)-s\right)\right] = 0 \quad \stackrel{\mathrm{d}/\mathrm{d}x}{\Longrightarrow} \quad V'(x)+\mu_1 f'(x) = 2\,⨍\,\mathrm{d}x'\,\frac{\rho(x')}{x-x'}$$

- Strategy : (*i*) solution is $\tilde\rho(x;\mu_1) \stackrel{\int\tilde\rho f = s}{\longrightarrow}$ (*ii*) get $\mu_1 = \mu_1^\star(s)$ and $\rho^\star(x;s) = \tilde\rho(x;\mu_1^\star(s))$
- Large deviation function : use $\dfrac{\mathrm{d}\mathscr{E}[\rho^\star(x;s)]}{\mathrm{d}s} = -\mu_1^\star(s)$ $\Rightarrow \Phi(s) = \int_s^{s_*}\mathrm{d}t\,\mu_1^\star(t)$ where $\mu_1^\star(s_*) = 0$.
- **Beyond the large deviation fct** : $\mathrm{d}\gamma_1\cdots\mathrm{d}\gamma_N\,\mathcal{P}_N(\cdots) \to \mathcal{D}\rho\, e^{-\frac{\beta N^2}{2}\mathscr{E}[\rho]+N(1-\frac{\beta}{2})S[\rho]}$ with $S[\rho] = -\int\rho\ln\rho$

Conjecture :

$$P_N^{(\beta)}(s) \underset{N\to\infty}{\simeq} c_{N,\beta}\sqrt{\frac{-\beta N^2}{4\pi}\frac{\partial\mu_1^\star(s)}{\partial s}}\,\exp\left\{-\frac{\beta N^2}{2}\int_s^{s_*}\mathrm{d}t\,\mu_1^\star(t) + N\left(1-\frac{\beta}{2}\right)S[\rho^\star]\right\} \qquad (6)$$

Several checks

- Laguerre & $f(x)=x$ (exact distrib. known) ✔
- Jacobi & $f(x)=x$ (conductance of QDs) ✔
  Mello & Baranger '99 ; ...
  Khoruzhenko, Savin & Sommers, PRB '09.
- Laguerre ($\theta=1$) & $f(x)=1/x$ (Wigner time delay in QDs) ✔
  compare with :
  * $N=1$: Gopar, Mello, Büttiker, PRL '97
  * $N=2$: Savin, Fyodorov, Sommers, PRE '01
  * large N: CT & Majumdar, PRL '13

Remark : $c_{N,\beta}$ unkown.

Application: $\tau_{\mathrm{W}} = s/N$

$$V_{\mathrm{eff}}(x) = x + \frac{\mu_1}{x} \Rightarrow \rho^\star(x;s) = \frac{x+c}{2\pi x^2}\sqrt{(x-a)(b-x)}$$

- Limit $\mu_1\to\infty$ ($s\to 0$) : $\rho^\star\to$ semi – circle

$a\simeq\frac{1}{s}-\sqrt{\frac{2}{s}}$, $\quad b\simeq\frac{1}{s}+\sqrt{\frac{2}{s}}$, $\quad c\simeq\frac{1}{s}$ & $\mu_1^\star(s)\simeq\frac{1}{s^2}$

$$\mathscr{E}[\rho^\star] \underset{s\to 0}{\simeq} \frac{1}{s}+\frac{1}{2}\ln s \quad \& \quad S[\rho^\star]\simeq -\frac{1}{2}\ln s$$

- Limit $\mu_1\to 0$ ($s\to\infty$) : $\rho^\star\to\rho_{\mathrm{MP}}$

$a\simeq\left(\frac{3}{4s}\right)^2$, $\quad b\simeq 4$, $\quad c\simeq 2a$ & $\mu_1^\star(s)\simeq\frac{27}{16s^3}$

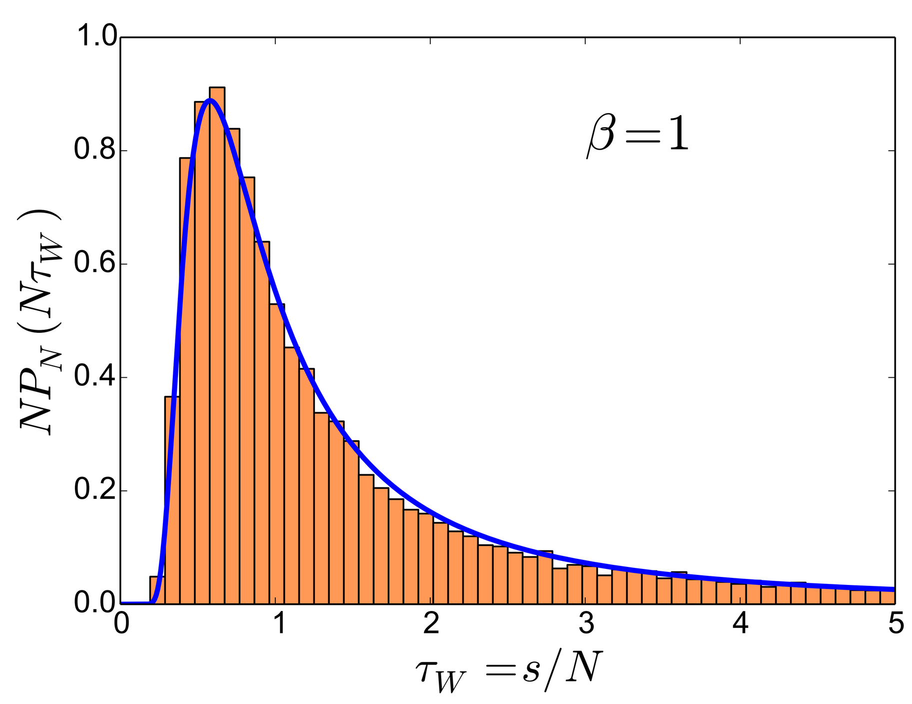
$$\mathscr{E}[\rho^\star] \underset{s\to 0}{\simeq} \mathscr{E}_{\mathrm{MP}} + \frac{27}{32\,s^2} \quad \& \quad S[\rho^\star]\simeq S_{\mathrm{MP}} + \frac{9(2-\sqrt{3})}{4\,s}$$

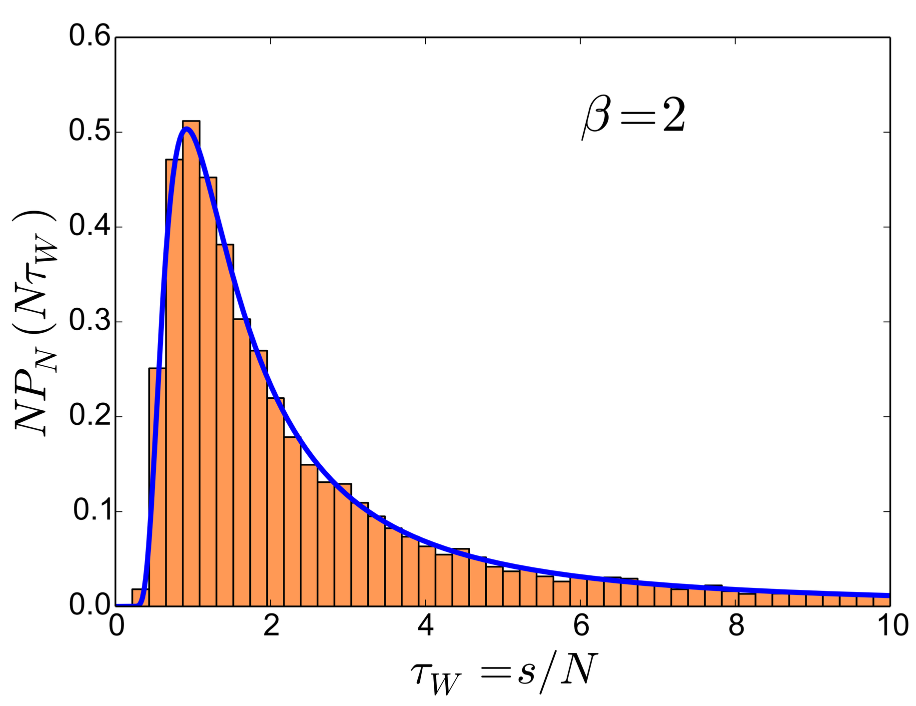
- Full crossover : ✔
- Scaling $s\sim N$ (DoS $\propto N$)

$$\mathscr{P}^{(\beta)}(\tau) = \lim_{N\to\infty} N\,P_N^{(\beta)}(s=N\tau) \quad \Rightarrow$$

Numerics

$N=100$:





Distribution of $\tau_{\mathrm{W}} = s/N$

$$\mathscr{P}^{(\beta)}(\tau) = \frac{C_\beta}{\tau^2}\exp\left\{-\frac{27\beta}{64\,\tau^2} + \left(1-\frac{\beta}{2}\right)\frac{9(2-\sqrt{3})}{4\,\tau}\right\}$$

References

- C. Texier & S. N. Majumdar, Phys. Rev. Lett. **110**, 250602 (2013) ; *ibid.* **112**, 139902 (2014) (E).
- A. Grabsch & C. Texier, J. Phys. A: Math. Theor. **49**, 465002 (2016).