Universality class for edge statistics associated with non-Hermitian random matrices

Oleg Zaboronski
(In collaboration with M. Poplavskyi and R. Tribe)

Department of Mathematics, University of Warwick

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Outline

Statement of the problem

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2. $\sqrt{N} + \lambda_{max}$ - the largest **real** eigenvalue
3. **Question:** the distribution of λ_{max} for $N \to \infty$, $\mathbb{P}[\lambda_{max} < t]$
4. **Closely related question**: the distribution of the right-most particle for annihilating Brownian motions (ABM's)

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2. **Forrester-Nagao (2007)**: The right tail of $\mathbb{P}[\lambda_{max} < t]$ is Gaussian
3. **Rider-Sinclair (2014)**: Exact representation for $\mathbb{P}[\lambda_{max} < t]$ in terms of a Fredholm determinant

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3. **Rider-Sinclair (2014)**: Exact representation for $\mathbb{P}[\lambda_{max} < t]$ in terms of a Fredholm determinant
4. **Garrod et al (2015)**: The edge scaling limit of the law of real eigenvalues for the real Ginibre ensemble coincides with the Pfaffian point process for the law of annihilating Brownian motions with step initial conditions

The main result

Theorem

For $t > 0$,

$$\mathbb{P}[\lambda_{max} < t] = 1 - \frac{1}{4} erfc(t) + O(e^{-2t^2}).$$

For $t < 0$,

$$\mathbb{P}[\lambda_{max} < t] = \exp\left(-\frac{\zeta(3/2)}{2\sqrt{2\pi}}|t| + O(1)\right).$$

Consequence for interacting particle systems

Corollary

Consider the system of instantaneously annihilating Brownian motions on the real line started with the step initial condition. Let $X_s^{(max)}$ *be the position of the rightmost particle at a fixed time* $s > 0$*. Then*

$$\mathbb{P}[X_s^{(max)} < x] = 1 - \frac{1}{4} erfc\left(\frac{x}{\sqrt{4s}}\right) + O\left(e^{-\frac{x^2}{2s}}\right)$$

for $x/\sqrt{s} \to \infty$*, while for* $x/\sqrt{s} \to -\infty$*,*

$$\mathbb{P}[X_s^{(max)} < x] = e^{\frac{1}{2\sqrt{2\pi}}\zeta\left(\frac{3}{2}\right)\frac{x}{\sqrt{4s}} + O(1)}.$$

The real Ginibre ensemble and annihilating Brownian motions

Theorem

Under the maximal entrance law, ABM's at $s > 0$ is a Pfaffian point process with the kernel $(4s)^{-1/2}K(y(4s)^{-1/2}, z(4s)^{-1/2})$, where

$$K(y,z) = -\frac{1}{2}\begin{pmatrix} F(y,z) & \partial_z F(y,z) \\ \partial_y F(y,z) & \partial_z\partial_y F(y,z) \end{pmatrix}$$

and $F(y,z) = 1 + \int_y^z \int_{-\infty}^y \frac{(u-v)}{\sqrt{2\pi}} e^{-\frac{(u-v)^2}{2}} \operatorname{erfc}\left(\frac{u+v}{\sqrt{2}}\right) du dv.$

- The same process describes the edge statistics of real eigenvalues for the $N \to \infty$ limit of the real Ginibre ensemble

From particles to random matrices

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3. **Forrester (2013)**: used the relation between Ginibre and $A + A \to \emptyset$ and Derrida-Zeitak's bulk gap formula for ABM's to conclude that

$$\mathbb{P}[G_\infty \text{ has no eigenvalues in } (a, a+D)] = e^{-\frac{1}{2\sqrt{2\pi}}\zeta(3/2)D+o(D)}$$

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4. As the width of the edge region for the spectrum of the real Ginibre is $O(1)$, it is reasonable to guess that the DZ formula also describes the left tail of the gap probability at the edge

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3. Fredholm Pfaffian (**Rains, 2000**):

$$\mathrm{Pf}(J + \lambda K_U) := \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \int_{U^k} dx_1 \ldots dx_k \mathrm{Pf}[K(x_i, x_j)]_{1 \le i,j \le k}.$$

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4. $(2,3) \Rightarrow \mathbb{P}[N_U = 0] = \mathrm{Pf}(J - K)$.

Determinantal representation for $\mathbb{P}[\lambda_{max} < t]$

Theorem (B. Rider and C. D. Sinclair, 2014)

Introduce the integral operator T with kernel

$$T(x,y) = \frac{1}{\pi}\int_0^\infty e^{-(x+u)^2} e^{-(y+u)^2} du.$$

Let χ_t be the indicator of (t,∞). Then,

$$\mathbb{P}[\lambda_{max} < t] = \sqrt{(1-a_t)}\sqrt{\det(I - T\chi_t)},$$

where $g(x) = \frac{1}{\sqrt{\pi}} e^{-x^2}$, $G(x) = \int_{-\infty}^{x} g(y)dy$ and $a_t = \int_t^\infty G(x)(I - T\chi_t)^{-1} g(x)dx$.

- Proof: use row-column manipulations similar to Tracy-Widom.

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- Let $(B_n, n \geq 0)$ be the discrete time random walk with Gaussian $N(0,1/2)$ increments.
- Then $T(x,y)$ is the probability density for a random walker starting at $(0,x)$ to finish at $(2,y)$ without visiting $\mathbb{R}_+$ at time $n=1$.

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Theorem

$$\mathbb{P}[\lambda_{max} < t] = \sqrt{\mathbb{P}(\tau_t < \tau_0)}e^{-\frac{1}{2}\mathbb{E}\left(\left(I_{\tau_0}-t\right)_+\delta_0\left(B_{\tau_0}\right)\right)},$$

where $x_+ := \max(x, 0)$ *is the positive part of a real number.*

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- Therefore, $a_t = \mathbb{P}(\tau_t > \tau_0)$, $1 - a_t = \mathbb{P}(\tau_t < \tau_0)$
- For $t < 0$, $a_t \sim O(1/t)$ (a martingale argument)

Computing the leading term for $t \to -\infty$

- The exponent is $e(t) = -\frac{1}{2}\mathbb{E}\left(\left(I_{\tau_0} - t\right)_+ \delta_0\left(B_{\tau_0}\right)\right)$

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$$\mathbb{E}(\delta(B_{\tau_0})) = \sum_{n=1}^{\infty} \mathbb{P}[\tilde{B}_1 < 0; \tilde{B}_2 < 0; \tilde{B}_{n-1} < 0 \mid \tilde{B}_n = 0]\mathbb{P}[\tilde{B}_n \in d0]$$

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- Therefore, $\mathbb{E}(\delta(B_{\tau_0})) = \frac{\zeta(3/2)}{\sqrt{2\pi}}$

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- The answer is universal within RMT (**Tau-Vu, 2015**)
- It also describes the left tail of the distribution of the rightmost particle for ABM's

Discussion

- At the macroscopic scale:
  $\Pr[G_N \text{ has no real e. v.'s}] = \exp\left(-\frac{\zeta(3/2)}{\sqrt{2\pi}}\sqrt{N} + o(\sqrt{N})\right)$.
  (**Poplavskyi et al, 2015**)

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  (**Poplavskyi et al, 2015**)
- Moreover, consider SEP with step initial condition. Let R_s be the position of the rightmost particle at time s. Then (**Krapivsky et al, 2015**),

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- Do gap statistics for the real Ginibre define a universality class of interacting particle systems? (Compare this with TW distribution and KPZ universality class.)

References

1. *What is the probability that a large random matrix has no real eigenvalues?* , arXiv Math.PR:1503.07926v1, (AAP, October 2016);

2. *On the distribution of the largest real eigenvalue for the real Ginibre ensemble*, arXiv Math.PR: 1603.05849 (to appear in AAP)

3. **Thank you!**