Random Fermionic Systems

Fabio Cunden Anna Maltsev Francesco Mezzadri

University of Bristol

December 9, 2016

Background

- First introduced to study magnetic properties of matter
- Toy model for quantum information – study of entanglement
- Random matrix aspect

Background

- First introduced to study magnetic properties of matter
- Toy model for quantum information – study of entanglement
- Random matrix aspect

Three papers that inspired this work:

- Lieb-Schultz-Mattis "Two soluble models of an Antiferromagnetic chain"
- Doctoral thesis of Huw Wells supervised by Jon Keating and Noah Linden, and subsequent work of Erdös and Schröder
- Keating-Mezzadri "Random Matrix Theory and Entanglement in Quantum Spin Chains"

Our object of study: the Hamiltonian

- Self-adjoint operator acting on $\mathbb{C}^{2^n}$
-

$$\mathcal{H} = \frac{1}{2}\sum_{i,j=1}^{n} A_{ij}({c_i}^\dagger c_j - c_i {c_j}^\dagger) + B_{ij}(c_i c_j - {c_i}^\dagger {c_j}^\dagger)$$

with $A_{ij} = A_{ji}, \quad B_{ij} = -B_{ij}$, i.e. $A = A^t$ and $B = -B^t$.

- c_j's are fermionic i.e. $\{c_i, c_j\} = 0, \{c_i, {c_j}^\dagger\} = \delta_{ij}$,

Our object of study: the Hamiltonian

- Self-adjoint operator acting on $\mathbb{C}^{2^n}$
- $$\mathcal{H} = \frac{1}{2}\sum_{i,j=1}^{n} A_{ij}(c_i{}^\dagger c_j - c_i c_j{}^\dagger) + B_{ij}(c_i c_j - c_i{}^\dagger c_j{}^\dagger)$$

  with $A_{ij} = A_{ji}, \quad B_{ij} = -B_{ij}$, i.e. $A = A^t$ and $B = -B^t$.
- c_j's are fermionic i.e. $\{c_i, c_j\} = 0, \{c_i, c_j{}^\dagger\} = \delta_{ij}$,

We take A_{ij}, B_{ij} iid real. Our conclusions:

Our object of study: the Hamiltonian

- Self-adjoint operator acting on $\mathbb{C}^{2^n}$
- $$\mathcal{H} = \frac{1}{2}\sum_{i,j=1}^{n} A_{ij}(c_i{}^\dagger c_j - c_i c_j{}^\dagger) + B_{ij}(c_i c_j - c_i{}^\dagger c_j{}^\dagger)$$

  with $A_{ij} = A_{ji}, \quad B_{ij} = -B_{ij}$, i.e. $A = A^t$ and $B = -B^t$.
- c_j's are fermionic i.e. $\{c_i, c_j\} = 0, \{c_i, c_j{}^\dagger\} = \delta_{ij}$,

We take A_{ij}, B_{ij} iid real. Our conclusions:

- Ground state energy gap $O(1/n)$ with explicit formula if Gaussian entries
- DOS – Gaussian universally, also for A, B band
- No repulsion – numerics

Alternative notation:

$\mathcal{H}$ **is a quadratic form, so write**

$$\mathcal{H} = ((\underline{c}^{\dagger})^t \ \underline{c}) \cdot M \cdot \begin{pmatrix} \underline{c}^t \\ \underline{c}^{\dagger} \end{pmatrix},$$

where M is the $2n \times 2n$ real symmetric matrix $M = \frac{1}{2}\begin{pmatrix} A & -B \\ B & -A \end{pmatrix}$.

Universality

- Gaussian DOS vastly universal

Universality

- Gaussian DOS vastly universal
- Subset sums: given a set $\{\lambda_1, ..., \lambda_n\}$ and $S_j \subset \{1, ..., n\}$, eigenvalues of $\mathcal{H}$ are closely related to $\sum_{k \in S_j} \lambda_k$.
- A lot of information
  - Gaussian DOS
  - Groundstate energy gap

Universality

- Gaussian DOS vastly universal
- Subset sums: given a set $\{\lambda_1, ..., \lambda_n\}$ and $S_j \subset \{1, ..., n\}$, eigenvalues of $\mathcal{H}$ are closely related to $\sum_{k \in S_j} \lambda_k$.
- A lot of information
    - Gaussian DOS
    - Groundstate energy gap
- Relation to sums of weighted binomial random variables
  – can take Fourier transform explicitly!

Fermionic systems: how they arise?

- n sites with spins that are linear combinations of σ^x and σ^y (no σ^z)
- nearest neighbor interaction – the XY model

Fermionic systems: how they arise?

- n sites with spins that are linear combinations of σ^x and σ^y (no σ^z)
- nearest neighbor interaction – the XY model
- the corresponding Hamiltonian is

$$\sum_{k=1}^{n} \sum_{a\in\{x,y\}} \sum_{b\in\{x,y\}} \alpha_{k,a,b}\sigma_k^a \sigma_{k+1}^b$$

- Here $\sigma_j^{(a)} = I_2^{\otimes(j-1)} \otimes \sigma^{(a)} \otimes I_2^{\otimes(n-j)}$

Fermionic systems: how they arise?

- n sites with spins that are linear combinations of σ^x and σ^y (no σ^z)
- nearest neighbor interaction – the XY model
- the corresponding Hamiltonian is

$$\sum_{k=1}^{n} \sum_{a\in\{x,y\}} \sum_{b\in\{x,y\}} \alpha_{k,a,b}\sigma_k^a\sigma_{k+1}^b$$

- Here $\sigma_j^{(a)} = I_2^{\otimes(j-1)} \otimes \sigma^{(a)} \otimes I_2^{\otimes(n-j)}$
- via Jordan-Wigner transformation, the Hamiltonian is equivalent to a quadratic form in fermionic variables

Jordan-Wigner transformation

- Maps a spin chain to a quadratic form in fermionic operators: allows for an exact solution
- In reverse: model a system of interacting fermions on a quantum computer

Jordan-Wigner details

- Raising and lowering operators $a_i^\dagger = \sigma_i^x + i\sigma_i^y$ and $a_i = \sigma_i^x - i\sigma_i^y$
- Can recover Pauli spin operators by $\sigma_j^x = (a_j^\dagger + a_j)/2$, $\sigma_j^y = (a_j^\dagger - a_j)/2$, $\sigma_j^z = (a_j^\dagger a_j - 1/2)$

Jordan-Wigner details

- Raising and lowering operators $a_i^\dagger = \sigma_i^x + i\sigma_i^y$ and $a_i = \sigma_i^x - i\sigma_i^y$
- Can recover Pauli spin operators by $\sigma_j^x = (a_j^\dagger + a_j)/2$, $\sigma_j^y = (a_j^\dagger - a_j)/2$, $\sigma_j^z = (a_j^\dagger a_j - 1/2)$
- Not fermionic

Jordan-Wigner details

- Raising and lowering operators $a_i^\dagger = \sigma_i^x + i\sigma_i^y$ and $a_i = \sigma_i^x - i\sigma_i^y$
- Can recover Pauli spin operators by $\sigma_j^x = (a_j^\dagger + a_j)/2$, $\sigma_j^y = (a_j^\dagger - a_j)/2$, $\sigma_j^z = (a_j^\dagger a_j - 1/2)$
- Not fermionic
  - Partly fermionic: $\{a_j, a_j^\dagger\} = 1, a_j^2 = (a_j^\dagger)^2 = 0$
  - Partly bosonic: $[a_j^\dagger, a_k^\dagger] = [a_j^\dagger, a_k^\dagger] = [a_j, a_k] = 0$
- For fermionic let

$$c_j = \exp\left(\pi i \sum_{k=1}^{j-1} a_k^\dagger a_k\right) a_j$$

$$c_j^\dagger = a_j^\dagger \exp\left(-\pi i \sum_{k=1}^{j-1} a_k^\dagger a_k\right).$$

c_j's and $c_j^\dagger$'s are fermionic: $\{c_j, c_k^\dagger\} = \delta_{kj}, \{c_j, c_k\} = \{c_j^\dagger, c_k^\dagger\} = 0$

Lieb-Schultz-Mattis Antiferromagnetic Chain '61

- $H_\gamma = \sum_j (1+\gamma)\sigma_j^x \sigma_{j+1}^x + (1-\gamma)\sigma_j^y \sigma_{j+1}^y$
- Hamiltonian is a quadratic form in Fermi operators and can be explicitly diagonialized

Lieb-Schultz-Mattis Antiferromagnetic Chain '61

- $H_\gamma = \sum_j (1+\gamma)\sigma_j^x \sigma_{j+1}^x + (1-\gamma)\sigma_j^y \sigma_{j+1}^y$
- Hamiltonian is a quadratic form in Fermi operators and can be explicitly diagonialized
- For free ends:

$$H_\gamma = \frac{1}{2}\sum_{j=1}^{n-1} c_j^\dagger c_{j+1} + \gamma c_j^\dagger c_{j+1}^\dagger + \text{hc}$$

- study long-range order in ground state

Lieb-Schultz-Mattis

If $\mathcal{H} = (\underline{c}^\dagger \ \underline{c}) \cdot M \cdot \begin{pmatrix} \underline{c} \\ \underline{c}^\dagger \end{pmatrix}$, with $M = \frac{1}{2}\begin{pmatrix} A & -B \\ B & -A \end{pmatrix}$ for XY model as before

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 & 1 \\ 1 & 0 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 & -1 \\ -1 & 0 & 1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 0 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & -1 & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & -1 & 0 \end{pmatrix}$$

and A, B can be explicitly diagonalized.

In the '61 paper,

- Complete set of eigenstates
- General expression for the order between any two spins involving a Green's function
- Short, intermediate, and long range order for various situations

Bipartite Entanglement

Setup: XY and XX models with a constant transversal magnetic field
Study: Entropy E_p of entanglement between subsystems

- Vidal et al. computed E_p numerically
- Jin and Korepin compute E_p for XX model using the Fisher-Hartwig conjecture, which gives the leading order asymptotics of determinants of certain Toeplitz matrices
- Keating and Mezzadri study asymptotics of entanglement of formation of ground state using RMT methods

Wells PhD thesis

Hamiltonians of the form

$$H_n = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \sum_{a=1}^{3} \sum_{b=1}^{3} \alpha_{a,b,j} \sigma_j^{(a)} \sigma_{j+1}^{(b)} \quad (1)$$

for any $\alpha_{a,b,j} \in \mathbb{R}$ random Gaussian (some universality possible)

Wells PhD thesis

Hamiltonians of the form

$$H_n = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \sum_{a=1}^{3} \sum_{b=1}^{3} \alpha_{a,b,j} \sigma_j^{(a)} \sigma_{j+1}^{(b)} \quad (1)$$

for any $\alpha_{a,b,j} \in \mathbb{R}$ random Gaussian (some universality possible)

Remarks:

- RMT introduced directly into the Hamiltonian, as the Hamiltonian is itself a random matrix
- Includes the σ^z so Jordan-Wigner does not yield a quadratic form
- Obtains a Gaussian density of states

Wells Numerics in the XY case

For a Hamiltonian of the form

$$H_n = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \sum_{a=1}^{2} \sum_{b=1}^{2} \alpha_{a,b,j} \sigma_j^{(a)} \sigma_{j+1}^{(b)} \quad (2)$$

- Eigenvalue repulsion in the full model and lack of repulsion in the random XY model
- Convergence to a Gaussian in the random XY model
- Numerical estimate of the error in the random XY model is on the order of $1/n$ where n is the number of cubits

Extension by Erdös and Schröder

- Arbitrary graphs with maximal degree $\ll$ total number of edges
  - Gaussian DoS
- p-uniform hypergraphs
  - Correspond to p-spin glass Hamiltonians acting on n distinguishable spin-1/2 particles
  - At $p = n^{1/2}$, phase transition between the normal and the semicircle
  - **quantum-classical** transition

Summary

Known:

- DoS, spectral gap in (deterministic) XY model
- DoS in a random neighbor-to-neighbor Hamiltonian with XYZ

Numerics:

- DoS in a random XY model
- Rate of convergence in the random XY model
- Lack of repulsion

We establish:

- DoS in general bilinear forms of fermionic operators
- spectral gap in special cases

Diagonalizing M

- Eigenvalue equation: $\frac{1}{2}\begin{pmatrix} A & -B \\ B & -A \end{pmatrix}\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \lambda \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$.
- Equivalent to: $\begin{cases} A\phi_1 - B\phi_2 = 2\lambda\phi_1, \\ B\phi_1 - A\phi_2 = 2\lambda\phi_2. \end{cases}$

Diagonalizing M

- Eigenvalue equation: $\frac{1}{2}\begin{pmatrix} A & -B \\ B & -A \end{pmatrix}\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \lambda \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$.
- Equivalent to: $\begin{cases} A\phi_1 - B\phi_2 = 2\lambda\phi_1, \\ B\phi_1 - A\phi_2 = 2\lambda\phi_2. \end{cases}$
- If $\psi_1 = \phi_1 - \phi_2$ and $\psi_2 = \phi_1 + \phi_2$, then $\begin{cases} (A+B)\psi_1 = 2\lambda\psi_2, \\ (A-B)\psi_2 = 2\lambda\psi_1. \end{cases}$

Diagonalizing M

- Eigenvalue equation: $\frac{1}{2}\begin{pmatrix} A & -B \\ B & -A \end{pmatrix}\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \lambda \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$.
- Equivalent to: $\begin{cases} A\phi_1 - B\phi_2 = 2\lambda\phi_1, \\ B\phi_1 - A\phi_2 = 2\lambda\phi_2. \end{cases}$
- If $\psi_1 = \phi_1 - \phi_2$ and $\psi_2 = \phi_1 + \phi_2$, then $\begin{cases} (A+B)\psi_1 = 2\lambda\psi_2, \\ (A-B)\psi_2 = 2\lambda\psi_1. \end{cases}$
- Note that $(A-B)^T = (A+B)$ and hence we get

$$\frac{1}{4}(A+B)^T(A+B)\psi_1 = \lambda^2\psi_1.$$

Diagonalizing M

- Eigenvalue equation: $\frac{1}{2}\begin{pmatrix} A & -B \\ B & -A \end{pmatrix}\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \lambda \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$.
- Equivalent to: $\begin{cases} A\phi_1 - B\phi_2 = 2\lambda\phi_1, \\ B\phi_1 - A\phi_2 = 2\lambda\phi_2. \end{cases}$
- If $\psi_1 = \phi_1 - \phi_2$ and $\psi_2 = \phi_1 + \phi_2$, then $\begin{cases} (A+B)\psi_1 = 2\lambda\psi_2, \\ (A-B)\psi_2 = 2\lambda\psi_1. \end{cases}$
- Note that $(A-B)^T = (A+B)$ and hence we get

$$\frac{1}{4}(A+B)^T(A+B)\psi_1 = \lambda^2\psi_1.$$

$$\boxed{\lambda \in \sigma(M) \iff \sqrt{\lambda^2} \text{ is singular value of } \frac{A+B}{2}.}$$

Need Hermiticity to get new Fermi operators

- Let U be the orthogonal matrix that diagonalizes M.
- Then U is a linear canonical transformation in the sense that

$$U = \begin{pmatrix} G & K \\ G^T & K^T \end{pmatrix} \qquad \begin{cases} GG^T + KK^T = I_n \\ GK^T + KG^T = 0_n, \end{cases} \tag{3}$$

and

$$UMU^T = \frac{1}{2} \begin{pmatrix} \Lambda & 0 \\ 0 & -\Lambda \end{pmatrix},$$

with $\Lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$, $\lambda_i \geq 0$.

Need Hermiticity to get new Fermi operators

- Let U be the orthogonal matrix that diagonalizes M.
- Then U is a linear canonical transformation in the sense that

$$U = \begin{pmatrix} G & K \\ G^T & K^T \end{pmatrix} \qquad \begin{cases} GG^T + KK^T = I_n \\ GK^T + KG^T = 0_n, \end{cases} \tag{3}$$

  and

$$UMU^T = \frac{1}{2}\begin{pmatrix} \Lambda & 0 \\ 0 & -\Lambda \end{pmatrix},$$

  with $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$, $\lambda_i \geq 0$.
- Let $\eta_k, {\eta_k}^\dagger$ operators defined by

$$\begin{pmatrix} \underline{\eta} \\ \underline{\eta}^\dagger \end{pmatrix} = U \begin{pmatrix} \underline{c} \\ \underline{c}^\dagger \end{pmatrix}.$$

- Because of (3), the η's are Fermi operators as well.
- $\mathcal{H} = \sum_{k=1}^n \lambda_k {\eta_k}^\dagger \eta_k + cI_{2^n}$ where $\lambda_k \geq 0$ and $c = -\frac{1}{2}\sum_{k=1}^n \lambda_k$.

Diagonalizing $\mathcal{H}$: Fermi basis

- η_j acts as a lowering operator for $\eta_j^\dagger \eta_j$ i.e. if $\eta_j^\dagger \eta_j |\psi\rangle = |\psi\rangle$ then $\eta_j^\dagger \eta_j \eta_j |\psi\rangle = 0$
- $\eta_j^\dagger$ acts as a raising operator for $\eta_j^\dagger \eta_j$

Diagonalizing $\mathcal{H}$: Fermi basis

- η_j acts as a lowering operator for $\eta_j^\dagger \eta_j$ i.e. if $\eta_j^\dagger \eta_j |\psi\rangle = |\psi\rangle$ then $\eta_j^\dagger \eta_j \eta_j |\psi\rangle = 0$
- $\eta_j^\dagger$ acts as a raising operator for $\eta_j^\dagger \eta_j$
- $\eta_j^\dagger \eta_j$'s commute so there exists a state $|\psi\rangle$ which is a simultaneous eigenstate

Diagonalizing $\mathcal{H}$: Fermi basis

- η_j acts as a lowering operator for $\eta_j^\dagger\eta_j$ i.e. if $\eta_j^\dagger\eta_j|\psi\rangle = |\psi\rangle$ then $\eta_j^\dagger\eta_j\eta_j|\psi\rangle = 0$
- $\eta_j^\dagger$ acts as a raising operator for $\eta_j^\dagger\eta_j$
- $\eta_j^\dagger\eta_j$'s commute so there exists a state $|\psi\rangle$ which is a simultaneous eigenstate
- By raising and lowering the state $|\psi\rangle$ in all possible combinations, can construct a set of 2^n orthonormal states which are simultaneous eigenstates of the $\eta_j^\dagger\eta_j$

Diagonalizing $\mathcal{H}$: subset sums

The spectrum of $\mathcal{H}$ is characterized as follows:

$$\boxed{x \in \sigma(\mathcal{H}) \iff \exists S \subset \{1, \ldots, n\} \text{ such that } x = c + \sum_{k \in S} \lambda_k.} \tag{4}$$

where $c = -\frac{1}{2} \sum_{k=1}^{n} \lambda_k$

Diagonalizing $\mathcal{H}$: subset sums

The spectrum of $\mathcal{H}$ is characterized as follows:

$$\boxed{x \in \sigma(\mathcal{H}) \iff \exists S \subset \{1, \dots, n\} \text{ such that } x = c + \sum_{k \in S} \lambda_k.} \quad (4)$$

where $c = -\frac{1}{2} \sum_{k=1}^{n} \lambda_k$

Key point in our methodology:

We work with the subset sum structure to glean information for a fixed sequence of λ_j's. Then we introduce the randomness on λ_j's as in RMT.

Ground state energy gap: important physical quantity, reflects how sensitive is the system to perturbations

Theorem 1

For A, B with iid Gaussian entries up to symmetry, the rescaled energy gap $\sqrt{2n/\sigma}\Delta$ converges in distribution to a random variable whose probability density function is

$$f(x) = (1+x)e^{-\frac{x^2}{2}-x}.$$

Ground state energy gap: important physical quantity, reflects how sensitive is the system to perturbations

Theorem 1

For A, B with iid Gaussian entries up to symmetry, the rescaled energy gap $\sqrt{2n/\sigma}\Delta$ converges in distribution to a random variable whose probability density function is

$$f(x) = (1+x)e^{-\frac{x^2}{2}-x}.$$

- $x_{2^n} = \sum_{j=1}^{n} \lambda_j$ and $x_{2^n-1} = \sum_{j=2}^{n} \lambda_j$ yielding that

$$\Delta := x_{2^n} - x_{2^n-1} = \lambda_1$$

Ground state energy gap: important physical quantity, reflects how sensitive is the system to perturbations

Theorem 1

For A, B with iid Gaussian entries up to symmetry, the rescaled energy gap $\sqrt{2n/\sigma}\Delta$ converges in distribution to a random variable whose probability density function is

$$f(x) = (1 + x)e^{-\frac{x^2}{2}-x}.$$

- $x_{2^n} = \sum_{j=1}^{n} \lambda_j$ and $x_{2^n-1} = \sum_{j=2}^{n} \lambda_j$ yielding that

$$\Delta := x_{2^n} - x_{2^n-1} = \lambda_1$$

- Recall that λ_j are singular values of $A + B$
- Result for smallest eigenvalue value of Wishhart matrices by Edelman
- Note that Δ is very large compared to mean spacing (O(1/n) instead of 2^{-n})

The relation with iid Bernoullis

Let x_j be the eigenvalues of $\mathcal{H}$. Then

$$x_j = \frac{1}{2}\sum_{k\in S_j}\lambda_k - \frac{1}{2}\sum_{k\in S_j^c}\lambda_k$$

for some $S_j \subset \{1, \ldots, n\}$.
Then

$$d\mu_n = \frac{1}{2^n}\sum_{j=1}^{2^n}\delta_{x_j} = \text{prob. meas. of } \sum_{j=1}^{n}\lambda_j(B_j - 1/2)$$

where B_j are n independent Bernoulli random variables.

The relation with iid Bernoullis

Let x_j be the eigenvalues of $\mathcal{H}$. Then

$$x_j = \frac{1}{2}\sum_{k\in S_j}\lambda_k - \frac{1}{2}\sum_{k\in S_j^c}\lambda_k$$

for some $S_j \subset \{1,\ldots,n\}$.
Then

$$d\mu_n = \frac{1}{2^n}\sum_{j=1}^{2^n}\delta_{x_j} = \text{prob. meas. of } \sum_{j=1}^{n}\lambda_j(B_j - 1/2)$$

where B_j are n independent Bernoulli random variables.
Fix the triangular array of $\lambda_{j,n}$ with $\lambda_j \ll \sqrt[4]{n}$ for all j

Consequences

- CLT: Lindenberg condition, so $\frac{1}{\sqrt{n}} d\mu_n \to \mathcal{N}(0, \frac{1}{n}\sum\lambda_j^2)$
- Can get a precise formula for the Fourier transform of $\frac{1}{\sqrt{n}} d\mu_n$.

Details

1. Lindenberg condition states:
   - variances σ_k are finite
   - $s_n^2 = \sum_{k=1}^n \sigma_k^2$
   - $\lim_{n\to\infty} \frac{1}{s_n^2} \sum_{k=1}^n \mathbb{E}\left[(X_k)^2 \cdot \mathbf{1}_{\{|X_k|>\varepsilon s_n\}}\right] = 0$
   - yields convergence to a Normal distribution with variance s_n for sequences of λ_j so that the maximum $< \sqrt[4]{n}$
   - will show that the condition on the max is satisfied with $\mathbb{P} \to 1$ as $n \to \infty$
   - a Berry-Esseen estimate yields an error of $1/\sqrt{n}$

Details

1. Lindenberg condition states:
   - variances σ_k are finite
   - $s_n^2 = \sum_{k=1}^n \sigma_k^2$
   - $\lim_{n\to\infty} \frac{1}{s_n^2} \sum_{k=1}^n \mathbb{E}\left[(X_k)^2 \cdot \mathbf{1}_{\{|X_k| > \varepsilon s_n\}}\right] = 0$
   - yields convergence to a Normal distribution with variance s_n for sequences of λ_j so that the maximum $< \sqrt[4]{n}$
   - will show that the condition on the max is satisfied with $\mathbb{P} \to 1$ as $n \to \infty$
   - a Berry-Esseen estimate yields an error of $1/\sqrt{n}$
2. For the computation of the Fourier transform :
   1. Fourier transform of $\frac{1}{\sqrt{n}}\lambda_j(B_j - 1/2)$ is $\cos\left(\frac{t\lambda_j}{2\sqrt{n}}\right)$
   2. Fourier transform of the DoS is then $\prod_j \cos\left(\frac{t\lambda_j}{2\sqrt{n}}\right)$

Random Matrix Theory

Have to show that $\lambda_n \leq \sqrt{n}$ when σ^2 of matrix entries is $1/N$

1. We study the case where A, B have k non-trivial diagonals, e.g. when $k = 1$ we recover the XY model
2. Band covariance matrices?
3. Our proof sketch:
   - Top singular value equals the operator norm
   - triangle inequality means that we can work with hermitian A, B instead of non-Hermitian M
   - For Hermitian A, $\lambda_n \leq \sup_i \sum_j |a_{ij}|$
   - This and union bounds enough for a large deviation estimate for $k \leq n^{1-\epsilon}$, including the XY model case i.e.
   - $\mathbb{P}(\lambda_n > n^{1/4-\delta}) \to 0$

Our Numerics

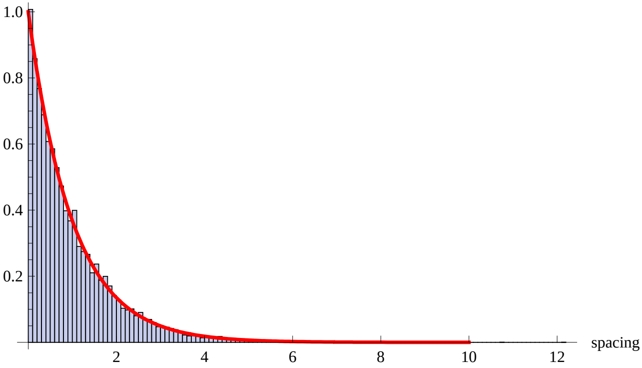

Figure: Spacing distribution for the unfolded spectrum.

Our Numerics

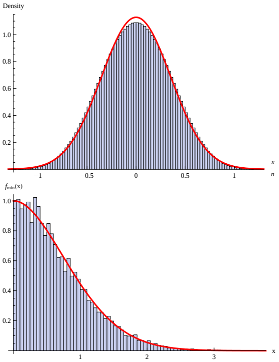

Figure: Density of states and ground state energy gap distribution for Gaussian quadratic form of Fermi operator. Here $n = 16$ (for a sample size of about 50).

Future study

Further questions we want to examine:

- Rate of convergence can probably be improved.
- The bottom eigenvalue of a band covariance matrix.
- In the bulk, the eigenvalues appear to form a Poisson process on the line.
- Speculation: relation to the Berry-Tabor conjecture. Generic integrable system $\Rightarrow$ Poisson statistics

Thank you!