Truncated linear statistics associated with the top eigenvalues of random matrices

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Problem

Starting from the jpdf of eigenvalues $P(\lambda_1, \cdots, \lambda_N)$, two well studied questions:

- distribution of the largest eigenvalue λ_1
- distribution of linear statistics $\sum_{n=1}^{N} f(\lambda_n)$

Here, intermediate problem: given any function f, determine the distribution of

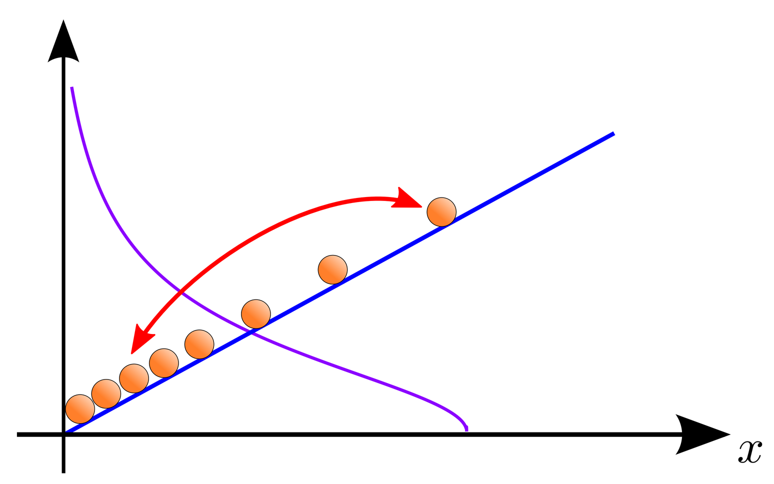
$$L = \sum_{n=1}^{\kappa N} f(\lambda_n)\,, \quad \lambda_1 > \lambda_2 > \cdots \lambda_N\,, \qquad (1)$$

for $0 < \kappa < 1$ fixed, in the limit $N \to \infty$.

Coulomb gas method

Density of eigenvalues: $\rho(x) = 1/N \sum_n \delta(x - x_n)$ has weight $P(\{x_n\}) \propto \exp(-N^2 \mathscr{E}[\rho])$, where

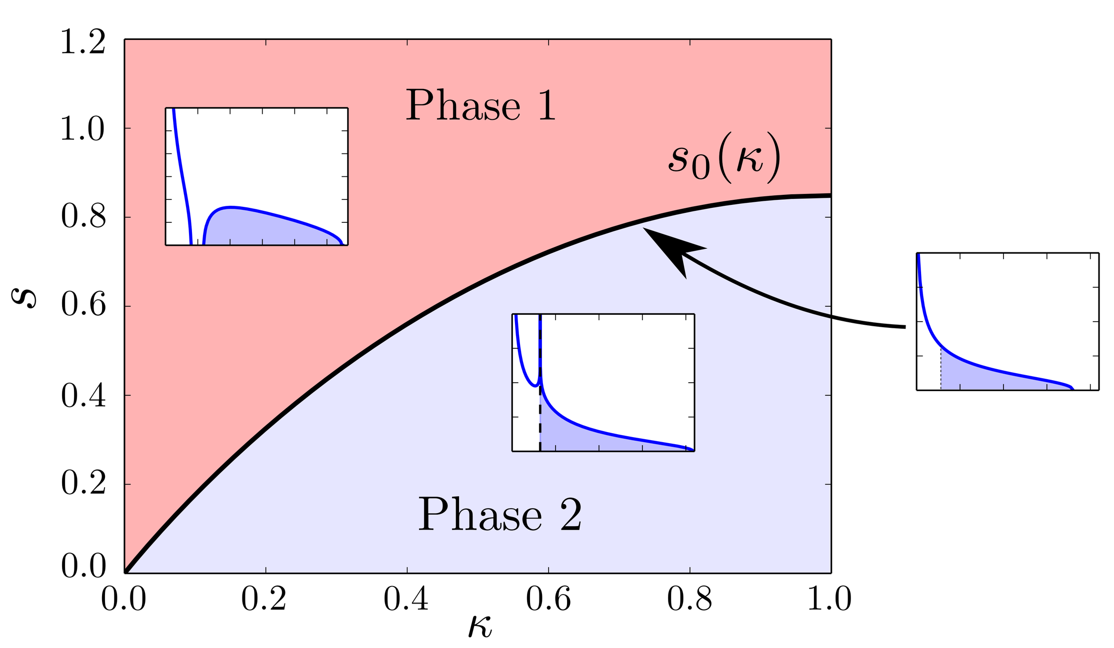
$$\mathscr{E}[\rho(x)] = \int \mathrm{d}x\, \rho(x)\, x - \int \mathrm{d}x \int \mathrm{d}y\, \rho(x)\rho(y)\, \ln|x-y|\,. \qquad (4)$$



Equilibrium density (minimum of $\mathscr{E}[\rho]$)
⇒ Marčenko-Pastur distribution

$$\rho_0^\star(x) = \frac{1}{2\pi}\sqrt{\frac{4-x}{x}}\,. \qquad (5)$$

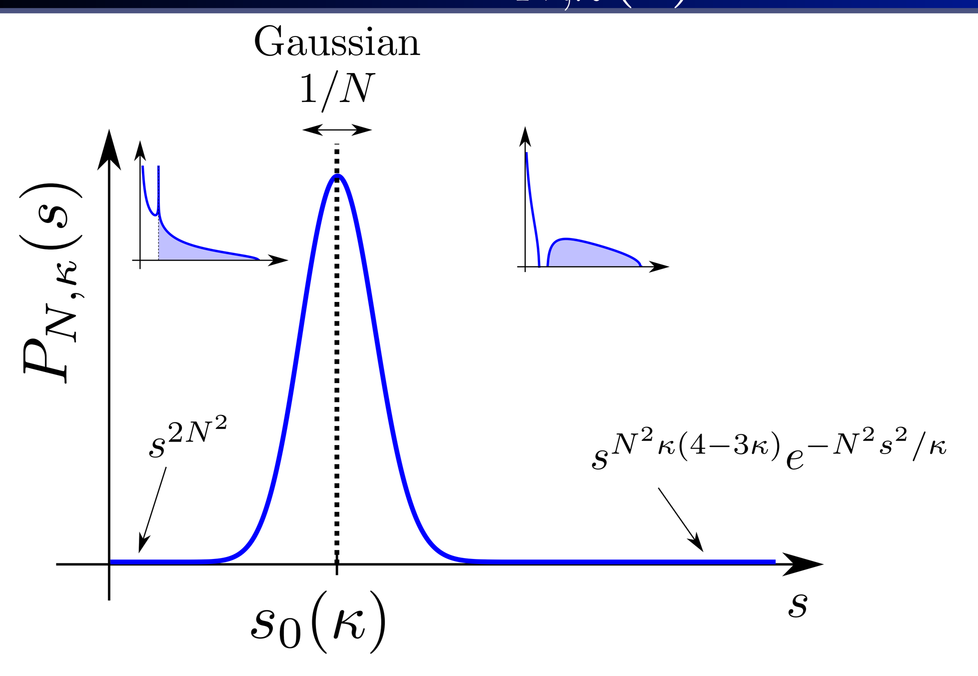
Phase diagram



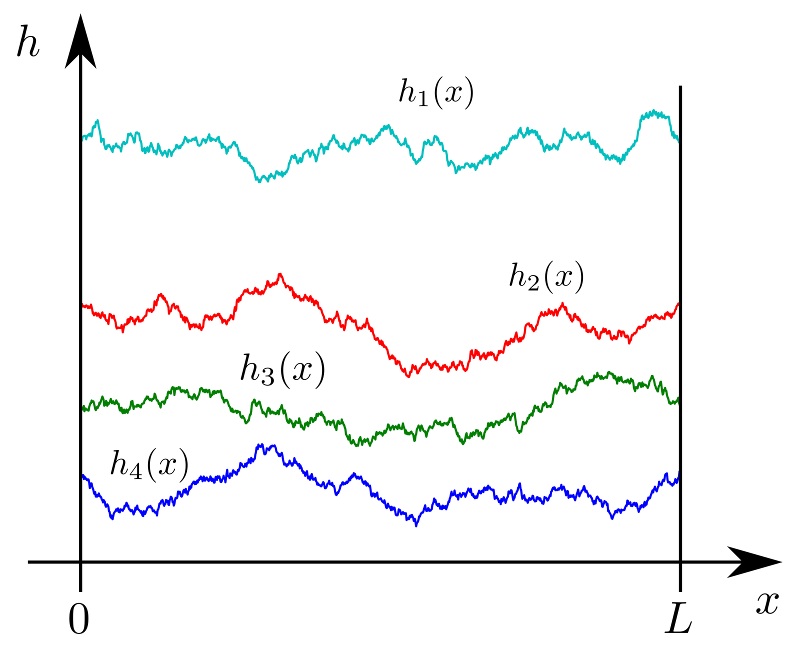
$\mathscr{E}[\rho]$ has an essential singularity at $s = s_0(\kappa)$
⇒ **Infinite order** phase transition

New universal mechanism, valid for all matrix ensemble and linear statistics

Distribution $P_{N,\kappa}(s)$



An example



Non-intersecting Brownian interfaces trapped in the confining potential $V(h) = b^2h^2/2 + \alpha(\alpha-1)/(2h^2)$.
Introduce $x_n = b\,h_n^2/N$. The jpdf of $\{x_n\}$ follow the Laguerre ensemble of RMT for $\beta = 2$ [1]:

$$P(x_1, \cdots, x_N) \propto \prod_{i<j}(x_i - x_j)^2 \prod_{n=1}^{N} x_n^{\alpha-1/2}\, \mathrm{e}^{-N x_n}\,. \qquad (2)$$

Center of mass of the κN highest interfaces (rescaled):

$$s = \frac{1}{N}\sum_{n=1}^{\kappa N}\sqrt{x_n}\,, \quad x_1 > x_2 > \cdots > x_N\,. \qquad (3)$$

Constrained Coulomb gas and large deviation function

The distribution $P_{N,\kappa}(s)$ is dominated by the configuration $\rho^\star(x;\kappa,s)$ which minimizes the energy under three constraints. Introduce

$$\mathscr{F}[\rho;\mu_0^{(1)},\mu_0^{(2)},\mu_1] = \mathscr{E}[\rho] + \mu_0^{(1)}\left(\int^c \rho(x)\mathrm{d}x - (1-\kappa)\right) + \mu_0^{(2)}\left(\int_c \rho(x)\mathrm{d}x - \kappa\right) + \mu_1\left(\int_c \sqrt{x}\rho(x)\mathrm{d}x - s\right).$$

Minimum of $\mathscr{F}$, $\delta\mathscr{F}/\delta\rho(x) = 0$:

$$2\int \rho^\star(y;\kappa,s)\ln|x-y|\,\mathrm{d}y = x + \begin{cases} \mu_0^{(1)} & \text{for } x < c \\ \mu_0^{(2)} + \mu_1\sqrt{x} & \text{for } x > c \end{cases} \qquad (6)$$

Then, the distribution $P_{N,\kappa}(s)$ is given by:

$$P_{N,\kappa}(s) \underset{N\to\infty}{\sim} \exp\left\{-N^2\Phi(\kappa;s)\right\}\,, \quad \text{where} \quad \Phi(\kappa;s) = \mathscr{E}[\rho^\star(x;\kappa,s)] - \mathscr{E}[\rho_0^\star(x)]\,. \qquad (7)$$

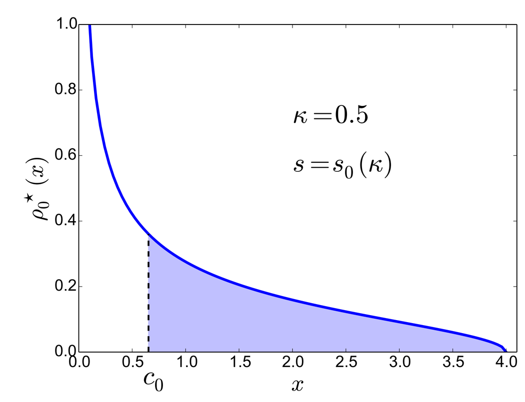
The energy is conveniently computed using the thermodynamic identity

$$\frac{\mathrm{d}\mathscr{E}[\rho^\star(x;\kappa,s)]}{\mathrm{d}s} = -\mu_1\,. \qquad (8)$$

Typical value $s_0(\kappa)$

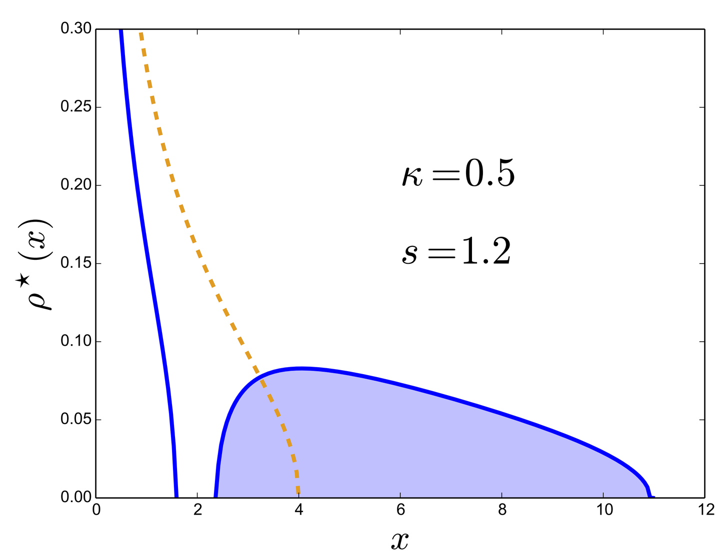
Typical value of $s_0(\kappa)$ taken by s (maximum of $P_{N,\kappa}(s)$) is given by $\mu_1 = 0$.
The corresponding density of eigenvalues is the Marčenko-Pastur law $\rho_0^\star(x)$.

$$\Rightarrow s_0(\kappa) = \int_{c_0}^{4}\rho_0^\star(x)\sqrt{x}\,\mathrm{d}x\,, \text{ where } c_0 \text{ is given by } \kappa = \int_{c_0}^{4}\rho_0^\star(x)\mathrm{d}x\,. \qquad (9)$$



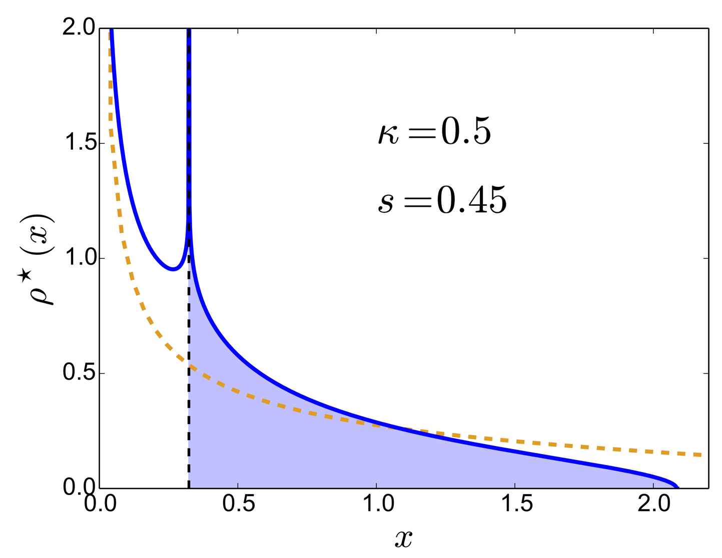
Phase 1: $s > s_0(\kappa)$

$$\rho^\star(x;\kappa,s) = \frac{\mu_1}{2\pi^2}\frac{\mathrm{sign}(x-c)}{\sqrt{d-b}}\sqrt{\frac{(c-x)(d-x)}{x(d-x)}} \times \Pi\left(\frac{d-c}{d-x}, \sqrt{\frac{d-c}{d-b}}\right).$$



Phase 2: $s < s_0(\kappa)$

$$\rho^\star(x;\kappa,s) = \frac{1}{2\pi}\sqrt{\frac{d-x}{x}} + \frac{\mu_1}{4\pi^2\sqrt{x}}\ln\frac{\sqrt{d-c}+\sqrt{d-x}}{|\sqrt{d-c}-\sqrt{d-x}|}\,.$$



→ **logarithmic divergence in the bulk**

References

[1] C. Nadal and S. N. Majumdar, Phys. Rev. E. **79**, 061117 (2009) → *full linear statistics* ($\kappa = 1$)

[2] A. Grabsch, S. N. Majumdar and C. Texier, preprint cond-mat arXiv:1609.08296