Time-delay for ballistic chaotic cavities: new results and a conjecture

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joint work with **Francesco Mezzadri** [1], **Nick Simm** [2] & **Pierpaolo Vivo** [3]

based on:

Phys. Rev. E **91**, 060102(R) (2015), arXiv:1412.2172
J. Phys. A: Math. Theor. **49**, 18LT01 (2016), arXiv:1601.06690
J. Math. Phys. **57**, 111901 (2016), arXiv:1607.00250

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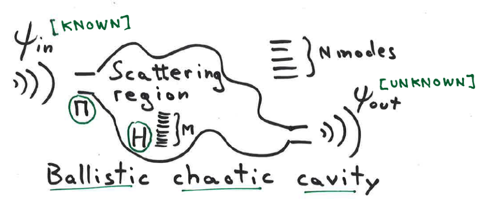
RMT approach to Quantum Transport



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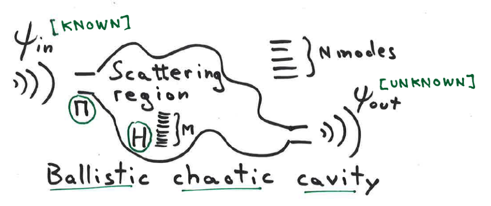

$$\boxed{\psi_{out} = S(E)\psi_{in}, \quad \text{where} \quad \psi_{in/out} \in \mathbb{C}^N, \; S(E) \in U(N)}$$

Scattering matrix: $S(E) = \dfrac{1 - iK(E)}{1 + iK(E)},$

Effective Hamiltonian: $K(E) = \Pi \dfrac{1}{E - H} \Pi^\dagger,$

Internal Hamiltonian and Coupling: $H = H^\dagger \quad (M \times M),$

$\Pi \quad (N \times M).$

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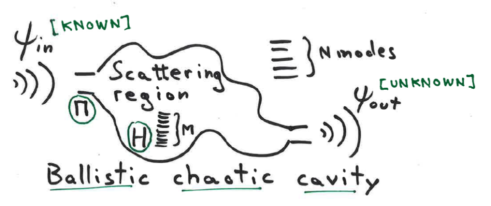

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$\Pi \quad (N \times M).$

RMT approach: $H \sim$ unitarily invariant ensemble of large size $(M \to \infty)$

Time-delay matrix

$$\boxed{Q = -i\hbar S^{\dagger}(E)\frac{\mathrm{d}S(E)}{\mathrm{d}E}}$$

$S(E)$: $N \times N$ scattering matrix; N: number of channels;
Conservation of probability: $S(E)$ unitary; Note that $Q = Q^{\dagger}$.

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Proper delay times

Eigenvalues of the Wigner-Smith matrix Q

$$\tau_1, \tau_2, \ldots, \tau_N$$

Wigner time delay

Time 'spent' by the particle in the scattering region

$$\mathrm{tr}Q = \tau_1 + \tau_2 + \cdots + \tau_N$$

Chaotic cavities

Time-delay matrix in chaotic regime with ideal couplings

Within the random-matrix ansatz, the proper delay times have joint law [Brouwer, Prahm and Beenakker (1997)]:

$$P(\tau_1, \cdots, \tau_N) \propto \prod_k f(\tau_k) \cdot \underbrace{\prod_{i<j} |\tau_i - \tau_j|^\beta}_{\text{repulsion}}$$

$f(\tau) = \theta(\tau)\tau^{\beta(1-\frac{3}{2}N)-2} \exp(-\beta\tau_H/2\tau)$; β: Dyson index; τ_H: Heisenberg time.

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Remark: The inverse $L = \frac{1}{N} Q^{-1}$ belongs to the Laguerre ensemble.

Let $\lambda_i = \frac{1}{N\tau_i} \geq 0$. The joint law of the inverse delay times is ($\tau_H = 1$)

$$\boxed{P(\lambda_1, \cdots, \lambda_N) \propto \prod_{i<j} |\lambda_i - \lambda_j|^\beta \prod_k \lambda_k^{\beta N/2} e^{-\beta N \lambda_k/2}}$$

Review of previous results

Despite the joint distribution for τ_i was explicitly known, the statistical properties of their sum (Wigner time delay) remained unknown for a while. The same for higher moments.

$$\boxed{\mathcal{T}_\kappa = N^{\kappa-1}\mathrm{tr}Q^\kappa}$$

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- Distribution of $\mathcal{T}_1$ for $N = 1$, Gopar, Mello and Büttiker (1996);
- Distribution of $\mathcal{T}_1$ for $N = 2$, Savin, Fyodorov and Sommers (2001);
- Averages $\langle \mathcal{T}_k \rangle$ for $N \gg 1$, Berkolaiko and Kuipers (2010);
- First large-N corrections to $\langle \mathcal{T}_k \rangle$ for $\beta = 1$ and 2, Berkolaiko and Kuipers (2010), Mezzadri and Simm (2012), Kuipers, Savin and Sieber (2014);
- Recursion for cumulants of $\mathcal{T}_1$ for all N, Mezzadri and Simm (2013);
- Averages and covariances of $\mathcal{T}_1$, $\mathcal{T}_2$ for $N \gg 1$, Grabsch and Texier (2014);
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- Representation for cumulants of $\mathcal{T}_k$ for $\beta = 2$ for all N, Novaes (2015);

How? RMT & semiclassical methods. For $\beta = 2$ orthogonal polynomials, representation theory of the symmetric group and connection with Painlevé III.

Large number of scattering channels N

$$\mathcal{T}_\kappa = N^{\kappa-1}\mathrm{tr}Q^\kappa$$

Cumulants (connected averages)

$$\langle \mathcal{T}_{\kappa_1} \cdots \mathcal{T}_{\kappa_\nu} \rangle_c \sim \left(\frac{1}{\beta N^2}\right)^{\nu-1} \alpha[\kappa_1, \ldots, \kappa_\nu]$$

$\nu = 1$: averages $\langle \mathcal{T}_\kappa \rangle_c \sim \alpha[\kappa]$;

$\nu = 2$: covariances $\langle \mathcal{T}_{\kappa_2} \mathcal{T}_{\kappa_2} \rangle_c \sim \dfrac{1}{\beta N^2}\alpha[\kappa_1, \kappa_2]$

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Main result: Generating functions

$$\boxed{F_\nu(z_1,\ldots,z_\nu) = \frac{1}{\beta^{\nu-1}}\sum_{\kappa_,\ldots,\kappa_\nu\geq 0}\alpha[\kappa_1,\ldots,\kappa_\nu]z_1^{\kappa_1}\cdots z_\nu^{\kappa_\nu}}$$

[FDC, PRE **91***, 060102(R) (2015), FDC, F. Mezzadri, N. Simm & P. Vivo, JPA* **49***, 18LT01 (2016)]*

Large number of scattering channels N

Theorem 1. The generating function of cumulants $\langle \mathcal{T}_{\kappa_1} \cdots \mathcal{T}_{\kappa_\nu} \rangle_c$ at leading order in N is

$$\boxed{F_\nu(z_1, \ldots, z_\nu) = (-1)^\nu z_1 \cdots z_\nu \, g_\nu(z_1, \ldots, z_\nu) + \delta_{1,\nu}(2 - z)]}$$

where the large-N ν-point connected resolvents

$$g_\nu(z_1, \ldots, z_\nu) = \lim_{N\to\infty} N^{2(\nu-1)} \left\langle \frac{1}{N} \sum_{i_1=1}^{N} \frac{1}{z_1 - \lambda_{i_1}} \cdots \frac{1}{N} \sum_{i_\nu=1}^{N} \frac{1}{z_\nu - \lambda_{i_\nu}} \right\rangle_c$$

can be computed recursively (in ν).

[FDC, Phys. Rev. E **91**, *060102(R) (2015)]*

[FDC, F. Mezzadri, N. Simm & P. Vivo, J. Phys. A: Math. Theor. **49**, *18LT01 (2016)]*

Large number of scattering channels N

$$F_v(z_1,\dots,z_v)=\frac{1}{\beta^{v-1}}\sum_{\kappa,\dots,\kappa_v\geq 0}\alpha[\kappa_1,\dots,\kappa_v]z_1^{\kappa_1}\cdots z_v^{\kappa_v}$$

Examples.

$$F_1(z_1)=\frac{1}{2}(3-z_1-\sqrt{z_1^2-6z_1+1}),$$

$$F_2(z_1,z_2)=\frac{1}{\beta}\frac{z_1z_2}{(z_1-z_2)^2}\left[\frac{z_1z_2-3(z_1+z_2)+1}{\sqrt{(z_1^2-6z_1+1)(z_2^2-6z_2+1)}}-1\right],$$

$$F_{3,0}(z_1,z_2,z_3)=\frac{16}{\beta^2}z_1z_2z_3\frac{[z_1z_2z_3-(z_1+z_2+z_3)+6]}{\prod_{i=1}^{3}(z_i^2-6z_i+1)^{3/2}}.$$

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$$F_1(z_1) = \frac{1}{2}(3 - z_1 - \sqrt{z_1^2 - 6z_1 + 1}),$$

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$$F_{3,0}(z_1,z_2,z_3) = \frac{16}{\beta^2} z_1 z_2 z_3 \frac{[z_1 z_2 z_3 - (z_1 + z_2 + z_3) + 6]}{\prod_{i=1}^{3}(z_i^2 - 6z_i + 1)^{3/2}}.$$

Remarks.

- algebraic functions (generic property of β-ensembles);
- easy to generate (see algorithm in FDC, F. Mezzadri, N. Simm & P. Vivo, JPA **49**, 18LT01 (2016)).

Sketch of the proof I

Working with the Laguerre ensemble $\lambda_i = \dfrac{1}{N\tau_i}$

$$\mathcal{T}_\kappa = N^{\kappa-1}\sum_i \tau_i^\kappa = \frac{1}{N}\sum_i \frac{1}{\lambda_i^\kappa}$$

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Key identity

$$\boxed{F_v(z_1, \ldots, z_v) = (-1)^v z_1 \cdots z_v \, g_v(z_1, \ldots, z_v), \qquad v \geq 2}$$

from Cauchy's integral formula:

$$\langle \mathcal{T}_{\kappa_1} \cdots \mathcal{T}_{\kappa_v} \rangle_c = \oint_{\mathcal{C}} \frac{\mathrm{d}z_1}{2\pi i} \cdots \oint_{\mathcal{C}} \frac{\mathrm{d}z_v}{2\pi i} \frac{g_v(z_1, \ldots, z_v)}{z_1^{\kappa_1} \cdots z_v^{\kappa_v}} \quad (\mathcal{C} \text{ does not enclose } z_i = 0)$$

$$= (-1)^v \oint_{|z_1|=\epsilon} \frac{\mathrm{d}z_1}{2\pi i} \cdots \oint_{|z_v|=\epsilon} \frac{\mathrm{d}z_v}{2\pi i} \frac{g_v(z_1, \ldots, z_v)}{z_1^{\kappa_1} \cdots z_v^{\kappa_v}} \quad (\text{for } \epsilon \text{ sufficiently small})$$

$$= (-1)^v \times \text{coefficient of } z_1^{\kappa_1 - 1} \cdots z_v^{\kappa_v - 1} \text{ in the power series of } g_v \text{ at } z_i = 0$$

Sketch of the proof II

Main ingredients for the recursion

1. The inverse delay times λ_i's form a one-cut β-ensemble.

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Sketch of the proof II

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3. At leading order in N a 'functional derivative method':

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4. Total Derivative Formula

$$\frac{\delta}{\delta V(z)} = \frac{\partial}{\partial V(z)} + \frac{\delta A}{\delta V(z)}\frac{\partial}{\partial A} + \frac{\delta B}{\delta V(z)}\frac{\partial}{\partial B} + \sum_{\ell \geq 1}\left\{\frac{\delta M_\ell}{\delta V(z)}\frac{\partial}{\partial M_\ell} + \frac{\delta \mathcal{J}_\ell}{\delta V(z)}\frac{\partial}{\partial \mathcal{J}_\ell}\right\}.$$

$\frac{\partial}{\partial V(z)}$: 'loop insertion operator';
A, B : edges of the cut;
$M_\ell, \mathcal{J}_\ell$: suitable coordinates [J. Ambjørn et al. (1993)].

Averages and covariances

Averages	κ	0	1	2	3	4	5	6	7	8	9	$\cdots$
	$\langle \mathcal{T}_\kappa \rangle_c \sim$	1	1	2	6	22	90	394	1806	8558	41586	$\cdots$

Known in combinatorics as the 'Large Schröder numbers'.

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Covariances

$$\langle \mathcal{T}_{\kappa_1} \mathcal{T}_{\kappa_2} \rangle_c \sim \frac{1}{\beta N^2} \begin{bmatrix} 4 & 24 & 132 & 720 & \cdots \\ 24 & 160 & 936 & 5312 & \\ 132 & 936 & 5700 & 33264 & \\ 720 & 5312 & 33264 & 198144 & \\ \vdots & \vdots & & & \ddots \end{bmatrix}$$

[FDC, Phys. Rev. E **91**, *060102(R) (2015)]*

Higher order cumulants

Table: A few values of cumulants $\langle \mathcal{T}_{\kappa_1}, \cdots \mathcal{T}_{\kappa_v} \rangle \sim (\beta N^2)^{1-v} \alpha[\kappa_1, \ldots, \kappa_v]$.

(κ_1)	$\alpha[\kappa_1]$	(κ_1, κ_2)	$\alpha[\kappa_1, \kappa_2]$	$(\kappa_1, \kappa_2, \kappa_3)$	$\alpha[\kappa_1, \kappa_2, \kappa_3]$
(1)	1	(1, 1)	4	(1, 1, 1)	96
(2)	2	(1, 2)	24	(1, 1, 2)	848
		(2, 2)	160	(1, 2, 2)	7488
				(2, 2, 2)	66112

$(\kappa_1, \kappa_2, \kappa_3, \kappa_4)$	$\alpha[\kappa_1, \kappa_2, \kappa_3, \kappa_4]$	$(\kappa_1, \kappa_2, \kappa_3, \kappa_4, \kappa_5)$	$\alpha[\kappa_1, \kappa_2, \kappa_3, \kappa_4, \kappa_5]$
(1, 1, 1, 1)	5088	(1, 1, 1, 1, 1)	437760
(1, 1, 1, 2)	54720	(1, 1, 1, 1, 2)	5303808
(1, 1, 2, 2)	569600	(1, 1, 1, 2, 2)	61526016
(1, 2, 2, 2)	5792256	(1, 1, 2, 2, 2)	690596352
(2, 2, 2, 2)	57876480	(1, 2, 2, 2, 2)	7553912832
		(2, 2, 2, 2, 2)	80925462528

[FDC, F. Mezzadri, N. Simm & P. Vivo, J. Phys. A: Math. Theor. **49**, *18LT01 (2016)]*

Further results and a conjecture

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Theorem 2. Let $\langle \mathcal{T}_{\kappa_1}, \cdots \mathcal{T}_{\kappa_\nu} \rangle \sim (\beta N^2)^{1-\nu} \alpha[\kappa_1, \ldots, \kappa_\nu]$.

(i) $\alpha[\kappa] \in \mathbb{N}$ (known in combinatorics as 'large Schröder numbers');

(ii) $\alpha[\kappa_1, \kappa_2] \in \mathbb{N}$;

(iii) $\alpha[\kappa_1, \kappa_2, \kappa_3] \in \mathbb{N}$;

(iv) $\alpha[\underbrace{1, \ldots, 1}_{\nu \text{ times}}] \in \mathbb{N}$ for all $\nu \geq 1$.

Proof by brute force.

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(i)+(ii)+(iii)+(iv) + numerical evidences suggest the following

Conjecture. $\alpha[\kappa_1, \ldots, \kappa_\nu] \in \mathbb{N}$.

[FDC, F. Mezzadri, N. Simm & P. Vivo, J. Phys. A: Math. Theor. **49**, *18LT01 (2016)]*

‘Genus’ expansion

Q: next-to-leading orders?

For generic β and N, let $\tau_k(N) = \langle \mathcal{T}_k \rangle$.

$$\boxed{\tau_k(N) = \sum_{g=0}^{\infty} \tau_{k,g} N^{-g}.}$$

The leading order is $\tau_{k,0} = \alpha[k]$ (independent of β).

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Finite-N formulae for $\tau_k(N)$ at $\beta = 2$:

$$\tau_k(N) = \frac{1}{k} \sum_{j=0}^{N-1} \binom{k+j-1}{k-1} \binom{k+j}{k-1} \frac{\Gamma(2N-k-j)}{\Gamma(N-j)} \frac{\Gamma(N+1)}{\Gamma(2N)} \quad \text{[Mezzadri-Simm'12]}$$

$$= \frac{1}{k!} \sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j} \frac{\Gamma(N-j+k)}{\Gamma(N-j)} \frac{\Gamma(N+j+1-k)}{\Gamma(N+j+1)} \quad \text{[Novaes'15].}$$

Both hard to extract large-N asymptotics from.
[First three terms $\tau_{k,0}, \tau_{k,2}, \tau_{k,4}$ of the asymptotics in Mezzadri-Simm ’12, Berkolaiko-Kuipers ’10, Kuipers-Savin-Sieber ’14].

‘Genus’ expansion

$$\tau_k(N) = \sum_{g=0}^{\infty} \tau_{k,g} N^{-g}.$$

Theorem 3. For $\beta = 2$, we have

$$\boxed{(N^2 - k^2)(k+1)\tau_{k+1} - 3N^2(2k-1)\tau_k + N^2(k-2)\tau_{k-1} = 0}$$

(an analog of Harer-Zagier recursion of GUE).

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The large-N coefficients $\tau_{k,g}$ satisfy the homogeneous linear recurrence

$$\boxed{(k+1)\tau_{k+1,2(g+1)} - 3(2k-1)\tau_{k,2(g+1)} + (k-2)\tau_{k-1,2(g+1)} = k^2(k+1)\tau_{k+1,2g}}$$

with initial conditions $\tau_{k,0} = \alpha[k]$, $\tau_{0,2g} = \delta_{0,2g}$, $\tau_{1,2g} = \delta_{0,2g}$.
All coefficients $\tau_{k,g}$ with odd g vanish identically.

[FDC, F. Mezzadri, N. Simm & P. Vivo, J. Math. Phys. **57***, 111901 (2016)]*

Generating functions

$$f_g(z) = \sum_{k=0}^{\infty} \tau_{k,g} z^k$$

Def: $y(z) = z^2 - 6z + 1$.

Corollary. If g is odd, $f_g(z) = 0$. For g even

$$\begin{cases} f_{g+2}(z) = \sqrt{y(z)} \int_0^z \frac{\mathrm{d}x}{y(x)^{3/2}} \left\{ x^2 f_g'''(x) + x f_g''(x) \right\}, \\ f_0(z) = \frac{3 - z - \sqrt{y(z)}}{2}. \end{cases}$$

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Remark. The recursion is systematic; $f_g(z)$, for $g = 0, \ldots, 4$, has been computed using semiclassical methods (Kuipers-Savin-Sieber '14).

Similar results for $\beta = 1$ (inhomogeneous recursions).

Methods similar to those of Haagerup-Thorbjoernsen for $\beta = 2$ and Ledoux for $\beta = 1$.

'Genus' expansion

$$\tau_k(N) = \sum_{g=0}^{\infty} \tau_{k,g} N^{-g}.$$

$(\beta = 2)$				g			
k	0	1	2	3	4	5	6
0	1	0	0	0	0	0	0
1	1	0	0	0	0	0	0
2	2	0	2	0	2	0	2
3	6	0	30	0	126	0	510
4	22	0	310	0	3262	0	31270
5	90	0	2730	0	57330	0	1048410
6	394	0	21980	0	805854	0	24848560

$(\beta = 1)$				g			
k	0	1	2	3	4	5	6
0	1	0	0	0	0	0	0
1	1	0	0	0	0	0	0
2	2	2	6	10	22	42	86
3	6	18	102	378	1638	6426	26214
4	22	128	1142	7048	47454	291696	1821094
5	90	840	10650	96000	904530	7786680	66945450
6	394	5306	89576	1092460	13529862	152881422	1704027412

‘Genus’ expansion and a conjecture

Numerical evidences suggest the following

Conjecture. $\tau_{k,g} \in \mathbb{N}$ for $\beta = 1$ and 2.

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Theorem 4. For $\beta = 2$: $\tau_{k,g} \in \mathbb{N}$ for $k \leq 10000$ or $g \leq 40$.

Indication of proof. We show that

$$\mathcal{J}_k(\zeta) = \sum_{g=0}^{\infty} \tau_{k,g}\zeta^g = \frac{P_k(\zeta^2)}{\prod_{j=0}^{k-1}(1 - j^2\zeta^2)}$$

where $P_k(\zeta)$ are polynomials satisfying a three term recursion. The denominator is a product of geometric series.

i) Compute $P_k(\zeta)$ using the recursion;

ii) Verify that $P_k(\zeta) \in \mathbb{N}[\zeta]$.

Use a symbolic algebra system. Same reasoning in the g ‘direction’.

The conjectures

$$F_v(z_1, \ldots, z_v) = \frac{1}{\beta^{v-1}} \sum_{\kappa, \ldots, \kappa_v \geq 0} \underbrace{\alpha[\kappa_1, \ldots, \kappa_v]}_{\text{integers ?}} z_1^{\kappa_1} \cdots z_v^{\kappa_v}$$

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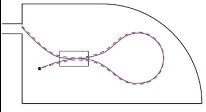

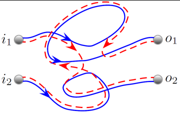

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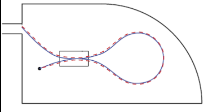

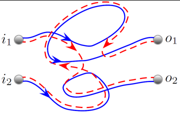

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If SM $\equiv$ RMT $\Rightarrow$ heuristic argument supporting the conjecture.

Take-home messages

- New results on the Wigner-Smith matrix of ballistic quantum dots: $\langle \mathrm{tr} Q^{\kappa_1} \cdots \mathrm{tr} Q^{\kappa_\nu} \rangle$ ($1/N$-expansion) in terms of generating functions;

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