# A Transition in Gap Probabilities - from 1 Gap to 2 Gaps in the Bulk

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- Let $K_s$ be the operator on $L^2(A)$ with the kernel

$$K_s(x,y) = \frac{\sin s(x-y)}{\pi(x-y)}.$$

- For a wide class of random matrices , the probability of finding no eigenvalues of a random matrix $M$ on a set $sA/\pi$ in the bulk scaling limit is given by the Fredholm determinant $\det(I - K_s)_A$.
- We focus on the asymptotic behaviour of $\det(I - K_s)_A$ as $s \to \infty$, in particular the probability of large gaps $sA/\pi$ where $A$ is composed of one or two intervals.

- As $s \to +\infty$,

$$\log \det(I - K_s)_{(\alpha,\beta)} = -\frac{(\beta-\alpha)^2 s^2}{8} - \frac{1}{4}\log s - \frac{1}{4}\log\frac{\beta-\alpha}{2} + c_0 + \mathcal{O}(s^{-1}), \qquad (1)$$

$$c_0 = \frac{1}{12}\log 2 + 3\zeta'(-1)$$

- Formula (1) was conjectured by des Cloizeaux and Mehta ('73) and Dyson ('76).
- The main term was rigorously proven by Widom ('94), and the second term by Deift, Its, Zhou in ('97).
- The constant $c_0$ was independently found by Krasovsky and Ehrhardt , with a further proof by Deift, Its, Krasovsky, Zhou ('04)-('07).

- For the 2-interval case we consider the determinant acting on

$$A = A_1 \bigcup A_2, \quad A_1 = (\alpha, -\nu), \;\; A_2 = (\nu, \beta).$$

- Let $R(z)$ have a branch cut on $A$ and be given by

$$R(z) = \sqrt{(z-\alpha)(z-\beta)(z^2-\nu^2)}, \quad R(z) \sim z^2 \text{ as } z \to \infty.$$

- Let $q(z) = z^2 + q_1 z + q_0$ be defined uniquely by

$$\int_{A_j} \frac{q(z)}{R_+(z)} dz = 0, \quad \text{for } j = 1, 2.$$

- Then as $z \to \infty$, we define $c_1$ by

$$\frac{q(z)}{R(z)} = 1 + \frac{c_1}{z^2} + \mathcal{O}(z^{-3}).$$

- For $z \in \mathbb{C}$ and $\Im\tau > 0$, the $\theta$-function is given by

$$\theta(z;\tau) = \sum_{m\in\mathbb{Z}} e^{2\pi i z m + \pi i \tau m^2}.$$

- Define the constants $V$ and $\tau$

$$V = -\int_{-\nu}^{\nu} \frac{q(z)}{R(z)} \frac{dz}{2\pi},$$

$$\tau = i\int_{-\nu}^{\nu} \frac{dz}{|R(z)|} \Big/ \int_{\nu}^{\beta} \frac{dz}{|R_+(z)|}.$$

- Widom (95) and Deift, Its, Zhou (97) showed that as $s \to \infty$,

$$\log\det(I - K_s)_A = -c_1 s^2 + \log\theta(sV;\tau) + G_1 \log s + G_2 + \mathcal{O}(s^{-1}),$$

where $G_1$ is explicitly given in terms of $\theta$-functions but $G_2$ is an unknown constant of integration.

- We have seen results for 1 interval and 2 intervals.
- The formula for 2 intervals held on

$$A = (\alpha, -\nu) \bigcup (\nu, \beta),$$

  for fixed $\alpha$, $\nu$ , $\beta$ as $s \to \infty$.
- We study the case where $\nu \to 0$ simultaneously as $s \to \infty$. We present results for the regime $s\nu \to 0$.

## Theorem (F, Krasovsky)

For the constant $\gamma = \frac{4}{\beta^{-1} - \alpha^{-1}}$, write $\nu$ in the form

$$\nu = \gamma e^{-\frac{s\sqrt{|\alpha\beta|}}{w}}, \quad w = k + x, \quad k \in \mathbb{N}, \ x \in [-1/2, 1/2).$$

Then, as $s \to \infty$ uniformly for $\nu \in (0, \nu_0)$, where $s\nu_0 \to 0$,

$$\log\det(I - K_s)_A = \log\det(I - K_s)_{(\alpha,\beta)}$$
$$+ s\sqrt{|\alpha\beta|}\left(w - \frac{x^2}{w}\right) + c(k) + \delta(x) + \mathcal{O}(\max\{s\nu_0, s^{-1}\}),$$

Let $G$ be the Barnes' $G$-function ($G(k+1) = \Gamma(k)G(k)$ and $G(1) = 1$). Then

$$c(k) = \log\left(\frac{2^{2k^2-k}}{\pi^k}\frac{G(k+1)^4}{G(2k+1)}\right).$$

- Alternatively, let $\kappa_j$ be the leading coefficients of the Legendre polynomials orthogonal on the interval $(-2, 2)$

$$c(k) = -\sum_{j=0}^{k-1} \log 2\pi\kappa_j^2$$

$$\kappa_j = 4^{-j-1/2}\sqrt{2j+1}\binom{2j}{j}.$$

- The function $\delta$ is given by

$$\delta(x) =$$

$$= \log\left(1 + 2\pi\kappa_{k-1}^2(\nu/\gamma)^{(2x+1)}\right) + \log\left(1 + (2\pi\kappa_k^2)^{-1}(\nu/\gamma)^{(1-2x)}\right).$$

## Theorem (F, Krasovsky)

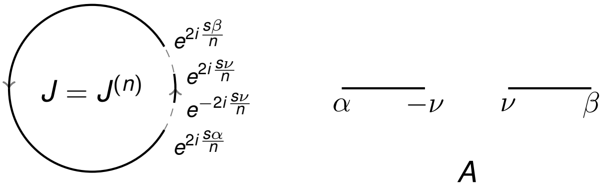
When $s \to \infty$ and $\nu \to 0$, in such a manner that $s\nu \to 0$ but $-s/\log\nu \to \infty$ the result reduces to

$$\log\det(I-K_s)_A = s^2\left(-\frac{(\beta-\alpha)^2}{8}+\frac{|\alpha\beta|}{\log(\gamma/\nu)}\right)-\frac{1}{2}\log s$$
$$+\frac{1}{4}\log\log(\gamma/\nu)-x^2\log(\gamma/\nu)+\log\left(1+(\nu/\gamma)^{1-2|x|}\right)$$
$$-\frac{1}{4}\log\left(\frac{\beta-\alpha}{2}\sqrt{|\alpha\beta|}\right)+2c_0+\mathcal{O}(\max\{s\nu_0,-1/\log\nu,-\frac{\log\nu}{s}\}).$$

- We have presented results for the scaling $s\nu \to 0$ as $s \to \infty$.
- It is work in progress to deal with $\nu$ such that $s\nu$ is bounded away from 0 and $s\nu \to \infty$ as $s \to \infty$.
- In bridging these regimes we expect to find that the fluctuations we see in the function $\delta$ to be caused by the degeneration of $\theta$-functions as $\nu \to 0$.
- It is interesting to note that the main order term in the work of Deift, Its, Zhou (for 2 intervals) matches our formula, but that the logarithmic term does not. This is not strange, as their results are only intended for fixed $\nu$.

- This transition from one gap to two gaps has an interesting parallel in the unitary random matrix ensembles exhibiting a transition from one-cut support to two-cut support as in the "birth of a cut", where asymptotic results for the correlation kernel where given simultaneously by Bertola & Lee, Claeys, Mo ('07)–('09). Fluctuations of the same type were also witnessed here.

# Method of proof




- Define the Toeplitz determinant on the interval $J$ by

$$D_n(\nu) = \det(f_{j-k})_{j,k=1}^{n}, \quad f_j = \int_J e^{-ij\theta}\frac{d\theta}{2\pi}$$

- We have the following link between the Toeplitz determinant and the Fredholm determinant

$$\lim_{n\to\infty} D_{n,s}(\nu) = \det(I - K_s)_A.$$

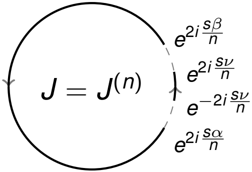
- Using the link with Toeplitz determinants, we reformulate the problem in terms of orthogonal polynomials on the unit circle.
- Specifically, we find a differential identity

$$\frac{\partial}{\partial \nu} \log D_{n,s}(\nu) = -\frac{1}{2\pi}[F_{n,s}(e^{-2i\frac{s\nu}{n}}) + F_{n,s}(e^{2i\frac{s\nu}{n}})]$$

where $F_{n,s}$ is given by

$$F_{n,s}(z) = -nY_{n+1,21}(0)|Y_{n,11}(z)|^2 + 2Y_{21,n+1}(0)\Re\left(z\overline{Y_{n,11}(z)}\frac{d}{dz}Y_{n,11}(z)\right).$$

- $Y$ is defined by orthogonal polynomials, but also solves an RH problem uniquely.




$Y$ uniquely solves the RH problem

(a) $Y : \mathbb{C} \setminus J \to \mathbb{C}^{2\times 2}$ is analytic;

(b) $Y$ has $L^2$ integrable boundary values on $J$:

$$Y_+(z) = Y_-(z) \begin{pmatrix} 1 & z^{-n} \\ 0 & 1 \end{pmatrix} \quad \text{for } z \in J$$

(c) $Y(z) = (I + \mathcal{O}(1/z)) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}$ as $z \to \infty$.

- We apply the Deift-Zhou steepest descent method to obtain asymptotics for $Y$ as $n \to \infty$, in the double scaling limit $s \to \infty$, $\nu \to 0$, $s/n \to 0$ and $s\nu \to 0$, evaluated at the points $e^{\pm 2i\frac{s\nu}{n}}$.
- Thus we obtain asymptotics for $\frac{\partial}{\partial \nu} \log D_{n,s}(\nu)$ and obtain asymptotics in the same double-scaling regime.
- Thereafter we integrate

$$\int_0^{\nu'} \frac{\partial}{\partial \nu} \log D_{n,s}(\nu) d\nu,$$

and use the identity between Toeplitz determinants and Fredholm determinants to obtain our results.