Subcritical multiplicative chaos for regularized counting statistics in the CUE

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Introduction

This poster is about the connection between random matrix theory and Gaussian multiplicative chaos (GMC). The GMC is a theory devoted to making sense of the exponential of a log-correlated Gaussian field. The basic quantity of interest is a centered Gaussian field $G(u)$ whose covariance is of the form

$$\mathbb{E}(G(u)G(v)) = \log\frac{1}{|u-v|} + T(u,v)$$

where T is a suitably well-behaved function. Because of the logarithmic singularity when $u = v$, $G(u)$ is not defined as an ordinary stochastic process, but can still be interpreted as a random distribution. Even so, this means the exponential of G is not well-defined in the usual way. Such objects are of considerable importance in mathematical physics and probability; a famous example is the case where G is the 2d Gaussian free field. Its exponential is conjectured to be the scaling limit of random planar maps in $2d$ quantum gravity.

Regularized fields and GMC

Since the origins of the subject of GMC, various ways were proposed to make mathematical sense of e^{G}. All of these ways give rise to a unique random measure μ_γ which can be formally written

$$\mu_\gamma(du) = e^{\gamma G(u) - \frac{\gamma^2}{2}\mathbb{E}(G(u)^2)}\, du \tag{1}$$

In physical contexts γ is often called the inverse temperature or intermittency parameter. A natural way to make sense of (1) is to take the convolution of G with a smooth mollifier. Let $\phi \geq 0$ be a smooth rapidly decaying function such that $\int_{\mathbb{R}} \phi(x)\, dx = 1$ and set

$$G_\epsilon(u) := (G \star \phi_\epsilon)(u) \tag{2}$$

where $\phi_\epsilon(u) = \frac{1}{\epsilon}\phi\left(\frac{u}{\epsilon}\right)$. Then $G_\epsilon(u)$ is a well-defined process and we get a well-defined measure $\mu_{\gamma,\epsilon}$ by replacing G with G_ϵ in (1). The following is the fundamental result underlying the theory of GMC.

Theorem 1 *Consider the sequence of random measures $\mu_{\gamma,\epsilon}(du)$ corresponding to the field $G_\epsilon(u)$. Then for any $w \in L^1(\mathbb{R})$ with compact support, we have the convergence in distribution $\mu_{\gamma,\epsilon}(w) \xrightarrow{d} \mu_\gamma(w)$ as $\epsilon \to 0$, where the limit variable $\mu_\gamma(w)$ is independent of the regularization ϕ and is non-trivial (i.e. not identically zero) if and only if $\gamma < \sqrt{2}$.*

Results of this flavour go back to the work of Kahane (1976) who was motivated by work of Mandelbrot on multifractality. A modern approach using the mollifiers (2) can be found in Robert and Vargas '10 or [1].

CUE and counting statistics

The CUE (*Circular Unitary Ensemble*) is the set of $N \times N$ unitary matrices U sampled with Haar measure. Let $e^{i\theta_1}, \ldots, e^{i\theta_N}$ be the eigenvalues of U with eigenangles $\theta_1, \ldots, \theta_N \in [0, 2\pi)$. We consider the random process which counts the number of eigenangles in a mesoscopic box:

$$X_N(u) = \sum_{j=1}^{N} \chi_u(N^\alpha \theta_j), \qquad \chi_u(\theta) := \pi 1_{\theta \in [u-l/2, u+l/2]}. \tag{3}$$

where the parameter $0 < \alpha < 1$ defines the mesoscopic regime and $l > 0$ is a length parameter. Throughout we will work with the following family of regularizations

$$X_{N,\epsilon}(u) = \sum_{j=1}^{N} (\chi_u \star \phi_\epsilon)(N^\alpha \theta_j), \qquad \phi_\epsilon(x) = \frac{1}{\epsilon}\phi\left(\frac{x}{\epsilon}\right) \tag{4}$$

and study the limit $N \to \infty$ with $\epsilon \to 0$ at a specified rate with N.

Soshnikov's CLT

For fixed $\epsilon > 0$ the regularization (4) is a smooth mesoscopic linear statistic and as such is known to satisfy a central limit theorem (CLT). In 2000, A. Soshnikov showed that for any smooth h with sufficient decay at $\pm\infty$ we have the CLT

$$\sum_{j=1}^{N} h(N^\alpha \theta_j) - \mathbb{E}\left(\sum_{j=1}^{N} h(N^\alpha \theta_j)\right) \xrightarrow{d} \mathcal{N}\left(0, \int_{\mathbb{R}} |k||\hat{h}(k)|^2\, dk\right) \tag{5}$$

The covariance structure can be written

$$\lim_{N\to\infty} \mathrm{Cov}\left(\sum_{j=1}^{N} h(N^\alpha\theta_j), \sum_{j=1}^{N} g(N^\alpha\theta_j)\right) = \int_{\mathbb{R}} |k|\hat{h}(k)\hat{g}(-k)\, dk = \frac{1}{2\pi^2}\int_{\mathbb{R}^2} h'(x)g'(y)\log\frac{1}{|x-y|}\, dx\, dy \tag{6}$$

The universality of this CLT has been the subject of considerable interest recently and has now been proved for a variety of random matrix ensembles (GUE, invariant ensembles, Wigner, β-ensembles etc.).

A limiting Gaussian field

Soshnikov's CLT cannot be applied directly to (4) because we want to consider $\epsilon \equiv \epsilon_N \to 0$ as $N \to \infty$. The variance diverges logarithmically in this limit

$$\mathrm{Var}(X_{N,\epsilon_N}(u)) = \log\frac{1}{\epsilon_N} + O(1), \qquad N \to \infty$$

Nevertheless, it is possible to surmise from (6) that after re-centering, the field (4) converges as $N \to \infty$ to a *generalized* stationary Gaussian field $G(u)$ with covariance structure

$$\mathbb{E}(G(u)G(v)) = \log\frac{1}{|u-v|} + \log\sqrt{|l^2 - (u-v)^2|}.$$

This field can also be written directly as a $1/f$ noise

$$G(u) = \mathrm{Re}\left\{\int_0^\infty \frac{\sin(\pi\kappa l)}{\sqrt{\kappa}} e^{-2\pi i\kappa u}\, dB_{\mathbb{C}}(\kappa)\right\} \tag{7}$$

where $B_{\mathbb{C}}(\kappa)$ is a complex Brownian motion. The GMC $\mu_\gamma(du)$ associated with this log-correlated field can be constructed as in Theorem 1.

Convergence in the subcritical phase

In terms of (4) we define the centered and normalized field

$$Y^{\gamma}_{N,\epsilon}(u) = \gamma(X_{N,\epsilon}(u) - \mathbb{E}(X_{N,\epsilon}(u))) - \frac{\gamma^2}{2}\mathrm{Var}(X_{N,\epsilon}(u)) \tag{8}$$

When ϵ depends on N, we always work in the regime where the following *slow decay hypothesis* holds:

$$N^{\alpha-1}/\epsilon_N = O(N^{-\delta}), \qquad \delta > 0. \tag{9}$$

This condition ensures that statistic (4) stays in the regime where the CLT (5) is 'approximately' valid. Our main result concerns the sequence of random measures

$$\mu_{N,\epsilon_N,\gamma}(du) = e^{Y^{\gamma}_{N,\epsilon_N}(u)}\, du$$

Theorem 2 (Lambert, Ostrovsky and Simm [2]) *Suppose condition (9) holds and consider (4) with $\phi \geq 0$ smooth, compactly supported and $\int_{\mathbb{R}} \phi(x)\, dx = 1$. For any $\gamma < \sqrt{2}$ and any $w \in L^1(\mathbb{R})$ with compact support, we have the convergence in distribution $\mu_{N,\epsilon_N,\gamma}(w) \xrightarrow{d} \mu_\gamma(w)$ as $N \to \infty$, where $\mu_\gamma(du)$ is the GMC measure associated with (7).*

Strong Gaussian approximation

A fundamental ingredient in our proofs is that fields such as (4) satisfy what we refer to as a *strong Gaussian approximation*:

$$\mathbb{E}\left[\exp\left(\sum_{k=1}^{q} t_k \overline{X}_{N,\epsilon_k}(u_k)\right)\right] = \exp\left(\frac{1}{2}\sum_{k,j=1}^{q} t_k t_j Q_{\epsilon_k,\epsilon_j}(u_k, u_j)\right)(1 + o(1)) \tag{10}$$

where $\overline{X}_{N,\epsilon}(u) = X_{N,\epsilon}(u) - \mathbb{E}(X_{N,\epsilon}(u))$ and $Q_{\epsilon_k,\epsilon_j}(u,v) \sim \log(\min\{|u-v|^{-1}, \epsilon_k^{-1}, \epsilon_j^{-1}\})$ as $|u-v| \to 0$. Here it is important that *the error bound is uniform for $u_1, \ldots, u_q$ varying in compact subsets of $\mathbb{R}$, subject to the slow decay* (9). With this estimate, it is fairly easy to prove Theorem 2 in the phase $\gamma < 1$, since

$$\mu_{\epsilon_N,N,\gamma}(w) = \mu_{\epsilon,N,\gamma}(w) + (\mu_{\epsilon_N,N,\gamma}(w) - \mu_{\epsilon,N,\gamma}(w)) \tag{11}$$

and when $\gamma < 1$ the L^2 norm of the blue term tends to zero in the consecutive limits $N \to \infty$ followed by $\epsilon \to 0$, while the red term converges in distribution to $\mu_\gamma(w)$ in the same limit. This is formally similar to the approach of Webb [4] applied in the macroscopic regime ($\alpha = 0$), and for obvious reasons, is known as *the L^2-phase*.

The phase $1 \leq \gamma < \sqrt{2}$

In this phase of γ one usually encounters difficulties, as the nice L^2 technique now causes the blue term in (11) to diverge as $N \to \infty$. Instead, we proceed according to [1], which treated exactly Gaussian regularizations. The idea is to examine the so-called *γ-thick points*, where the field is close to its maximum. By exploiting the strong Gaussian approximation, we show that when $\tau > \gamma$, the quantity

$$\mu_{N,\epsilon_N,\gamma}\left(u \in [-c,c] : \overline{X}_{N,\epsilon}(u) > \tau\log(\epsilon^{-1}) \quad \forall \epsilon \in \{e^{-k} : L \leq k \leq \log\frac{1}{\epsilon_N}\}\right)$$

converges to zero in L^1 as $N \to \infty$ followed by $L \to \infty$. Then it is sufficient to examine the quantity

$$\mu_{N,\epsilon_N,\gamma}\left(u \in [-c,c] : \overline{X}_{N,\epsilon}(u) \leq \tau\log(\epsilon^{-1}) \quad \forall \epsilon \in \{e^{-k} : L \leq k \leq \log\frac{1}{\epsilon_N}\}\right)$$

for which an L^2 estimate continues to apply, even when $1 \leq \gamma < \sqrt{2}$ and τ is sufficiently close to γ. Then a proof of convergence can be established by appropriately decomposing the blue term in (11) according to these events.

Moments and Selberg's integral

The total mass of the field (8) on an interval, say $[0,r]$, is defined as $\mu_{N,\epsilon,\gamma}[0,r]$. One of our initial motivations for establishing (10) is that we can use it to compute moments of all orders $q \in \mathbb{N}$.

Theorem 3 (Lambert, Ostrovsky and Simm [2]) *Suppose that the hypotheses of Theorem 2 are satisfied. Furthermore let $l = L_N = N^\beta$ with $0 < \beta < \alpha$ and suppose $\gamma^2 q < 2$. We have the limits*

$$\lim_{N\to\infty} L_N^{-\gamma^2\binom{q}{2}}\mathbb{E}(\mu_{N,\epsilon_N,\gamma}[0,r])^q = \lim_{l\to\infty}\lim_{\epsilon\to 0}\lim_{N\to\infty} l^{-\gamma^2\binom{q}{2}}\mathbb{E}(\mu_{N,\epsilon,\gamma}[0,r])^q$$
$$= \int_{[0,r]^q} |\Delta(\mathbf{u})|^{-\gamma^2}\, d\mathbf{u} = r^{q+\gamma^2\binom{q}{2}}\prod_{j=0}^{q-1}\frac{\Gamma(1-(j+1)\gamma^2/2)\Gamma^2(1-j\gamma^2/2)}{\Gamma(1-\gamma^2/2)\Gamma(2-(q+j-1)\gamma^2/2)}.$$

where $\Delta(\mathbf{u}) = \prod_{1\leq j<k\leq q}(u_k - u_j)$.

Thie equivalence of these limits with Selberg's integral proves the particular case $q \in \mathbb{N}$ of the conjectures in [3], where the single limit in N is called the *strong conjecture* while the triple limit is the *weak conjecture*.

Proving the strong Gaussian approximation

The proof of an approximation like (10) is an interesting problem in random matrix theory in its own right and goes beyond the CLT (5). For the CUE, our main tool for proving this estimate is the *Borodin-Okounkov formula*, which re-writes the the left-hand side of (10) *exactly* in terms of the Laplace transform of a (N-dependent) Gaussian multiplied by a Fredholm determinant. The latter determinant is essentially the error term we want to estimate uniformly, which can be done precisely in the regime (9). Hence the meaning of this condition is that we remain in the 'strong Szegő regime' where the CLT (5) holds approximately with a diverging variance. If $\epsilon_N \to 0$ faster than this, we enter the regime of merging Fisher-Hartwig singularities where the associated uniformity problem is much more delicate. This is the essential reason we consider the regularized counting statistics (4).

Conclusions and universality

A key message of this poster is that many natural quantities appearing in random matrix theory (such as the counting statistics (3) or the log of the characteristic polynomial) behave like a regularization of some log-correlated Gaussian field G. An interesting feature of these regularizations is that unlike (2) they are *non-Gaussian* for finite N. Hence, in the context of GMC we have studied the measures (1) with G replaced with such random matrix regularizations and the limit $N \to \infty$. Our main result is to show that the analogue of Theorem 1 is true for the CUE in the whole subcritical phase $\gamma < \sqrt{2}$.

In GMC, the different random matrix models may be interpreted as different ways of regularizing a log-correlated field and it's natural to expect that the limit is independent of the regularization. The counterpart of this idea in random matrix theory is of course *universality*. To add some weight to these claims, we have also given a proof of Theorem 2 (in fact, a somewhat stronger result) for the *sine process* which captures the universal features of large classes of random matrices, see [2] for further details.

References

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