Limiting spectral distribution of random band matrices

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Abstract

We consider the limiting spectral distribution of matrices of the form $\frac{1}{2b_n+1}(R+X)(R+X)^*$, where X is an $n\times n$ random band matrix of bandwidth b_n and R is a non random band matrix of bandwidth b_n. We show that the Stieltjes transform of spectrum of such matrices converges to the Stieltjes transform of a non-random measure. And the limiting Stieltjes transform satisfies an integral equation. For $R=0$, the integral equation yields the Stieltjes transform of the Marchenko-Pastur law.

Definition [Periodic band matrix]

An $n\times n$ matrix $M=(m_{ij})_{n\times n}$ is called a periodic band matrix of bandwidth b_n if $m_{ij}=0$ whenever $b_n<|i-j|<n-b_n$.

M is called a non-periodic band matrix of bandwidth b_n if $m_{ij}=0$ whenever $b_n<|i-j|$.

[Poincaré inequality]

Let X be a $\mathbb{R}^k$ valued random variable with probability measure μ. The probability measure μ is said to satisfy the Poincaré inequality with constant $m>0$, if for all continuously differentiable functions $f:\mathbb{R}^k\to\mathbb{R}$,

$$\mathrm{Var}(f(X))\le \frac{1}{m}\mathbb{E}(|\nabla f(X)|^2).$$

Construction

Let $X=(x_{ij})_{n\times n}$ be an $n\times n$ periodic band matrix of bandwidth b_n, where $b_n\to\infty$ as $n\to\infty$. Let R be a sequence of $n\times n$ deterministic periodic band matrices of bandwidth b_n. Let us denote $c_n=2b_n+1$ and μ_M be the ESD of M. Assume that

(a) $\mu_{\frac{1}{c_n}RR^*}\to H$, for some non random probability distribution H

(b) $\{x_{jk}:\ k\in I_j,\ 1\le j\le n\}$ is an iid set of random variables,

(c) $\mathbb{E}[x_{11}]=0, \mathbb{E}[|x_{11}|^2]=1,$

and define (d) $Y=\frac{1}{\sqrt{c_n}}(R+\sigma X)$, where $\sigma>0$ is fixed.

$$\tag{1}$$

Remarks

- ✓ The theorem is also true for non-periodic band matrices.
- ✓ If we take $R=0$ and $\sigma=1$, then H is supported only at the real number 0. In that case (2), becomes
$$m(z)(1+m(z))z+1=0,$$
which is the same quadratic equation satisfied by the Stieltjes transform of Marchenko-Pastur law.

Theorem

Let Y be the band matrix as defined in (1). In addition to the existing assumption, assume that

(i) $\frac{n}{c_n^2}\to 0,$

(ii) H is compactly supported

(iii) $\mathbb{E}[|x_{11}|^{4p}]<\infty$, for some $p\in\mathbb{N}$.

Then $\mathbb{E}|m_n(z)-m(z)|^{2p}\to 0$ uniformly for all $z\in\{z:\Im(z)>\eta\}$ for any fixed $\eta>0$, and the Stieltjes transform of μ satisfies

$$m(z)=\int_{\mathbb{R}}\frac{dH(t)}{\frac{t}{1+\sigma^2 m(z)}-(1+\sigma^2 m(z))z}\quad \text{for any } z\in\mathbb{C}^+. \tag{2}$$

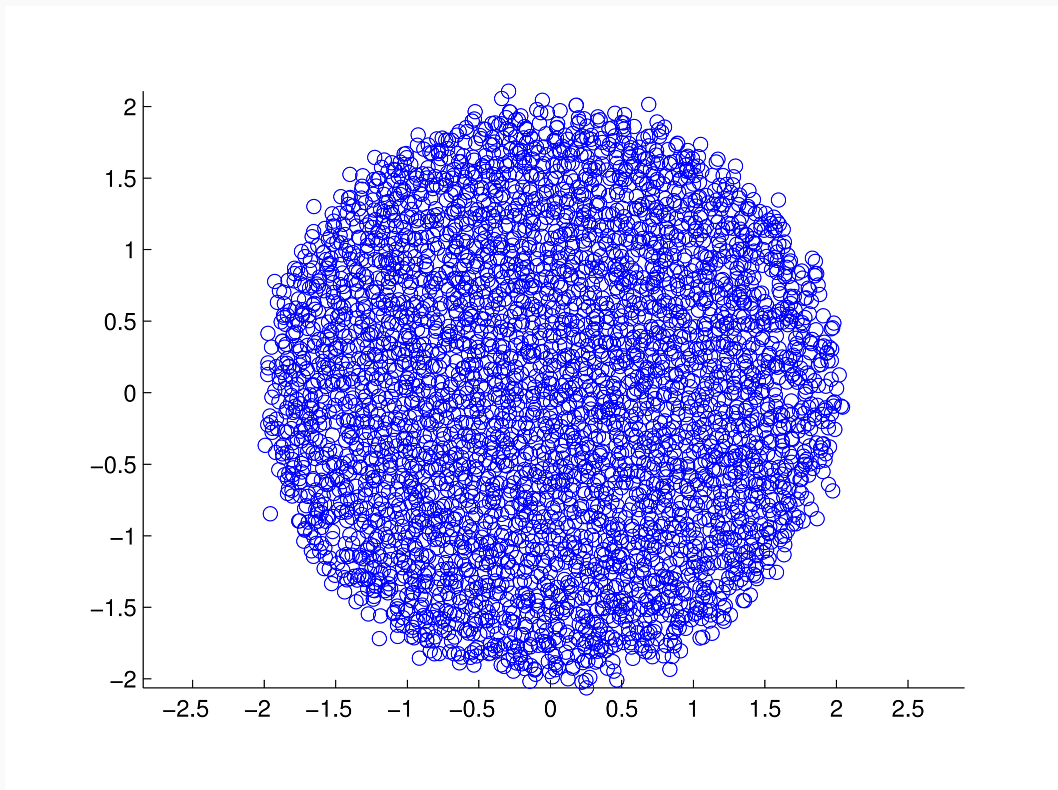
Remarks

- ✓ If $c_n=n^\beta$ where $\beta=\frac{1}{2}+\frac{1}{2p}$, then the convergence in Theorem is almost sure.
- ✓ The condition $c_n>>\sqrt{n}$ can be relaxed to $c_n=\Omega((\log n)^2)$. But then, we need the entries of the matrix to satisfy the Poincaré inequality.
- ✓ If $c_n=n^\alpha$, where $\alpha>0$, then the convergence in Theorem (with Poincaré assumption) is almost sure.

Discussion

- ✓ The real part of the Stieltjes transform $m_n(z):=\frac{1}{n}\sum_{i=1}^n\frac{1}{\lambda_i-z}$ can be written as
$$\begin{aligned}m_{nr}(z)&:=\Re(m_n(z))\\&=\frac{1}{n}\sum_{i=1}^n\frac{\Re(\lambda_i-z)}{|\lambda_i-z|^2}\\&=-\frac{1}{2}\frac{\partial}{\partial\Re(z)}\int_0^\infty \log x\,\nu_n(dx,z),\end{aligned}$$
where $\nu_n(\cdot,z)$ is the ESD of $(\frac{1}{\sqrt{n}}X_n-zI)(\frac{1}{\sqrt{n}}X_n-zI)^*$. And secondly the characteristic function of $\frac{1}{\sqrt{n}}X_n$ satisfies [1, section 1]
$$\int\int e^{i(ux+vy)}\mu_n(dx,dy)=\frac{u^2+v^2}{i4\pi u}\int\int\frac{\partial}{\partial s}\left[\int_0^\infty \log x\,\nu_n(dx,z)\right]e^{i(us+vt)}\,dtds,$$
for any $uv\neq 0$, where $z=s+it$ and $\mu_n(\cdot,\cdot)$ is the ESD of $\frac{1}{\sqrt{n}}X_n$.
- ✓ So, finding the limiting behaviour of $\nu_n(\cdot,z)$ is an essential ingredient in finding the limiting behaviour of $\mu_n(\cdot,\cdot)$.
- ✓ This idea was used in the proof of circular law by Tao et. al. [3].

Circular Law



Spectral distribution of a 4000×4000 random band matrix with bandwidth $\sqrt{4000}$ and i.i.d complex Gaussian entries.

Idea of the Proof

The main idea is inspired from Silverstein's work [2]. However, Band Random Matrices lack several symmetry properties of a full matrix. To get around that problem, we have found some concentration inequalities regarding the partial traces of resolvent of Band Random Matrices.

References

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