On the Optimum Asymptotic Multiuser Efficiency of Randomly Spread CDMA

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Consider a linear vector channel

\[ y = \frac{1}{\sqrt{N}} H b + n \]

with

\[ b \in \{ \pm 1 \}^K \]
\[ n \sim N(0, \sigma^2 I) \]

and \( H \) being an \( N \times K \) random matrix with iid entries of unit variance and vanishing odd moments.
Multiuser efficiency measures bitwise error probability.

\[ \eta \in [0; 1] \]
For unit variance Gaussian data, the multiuser efficiency is trivially related to the Stieltjes transform $G(s)$ in the large matrix limit:

$$\eta(\sigma^2) = \lim_{s \to \sigma^2} s G(-s).$$
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In general, the multiuser efficiency depends on both the eigenvalue spectrum of the random matrix $HH^\dagger$ and the distribution of the data.
Let $N, K \to \infty$, but $\beta = K/N$ fixed. Then, the multiuser efficiency with optimal detection is a solution to the fixed point equation

$$\frac{1}{\eta} = 1 + \frac{\beta}{\sigma^2} \left[ 1 - \sqrt{\frac{\eta}{2\pi\sigma^2}} \int_{\mathbb{R}} \tanh \left( \frac{\eta}{\sigma^2} x \right) \exp \left( -\frac{\eta(x - 1)^2}{2\sigma^2} \right) dx \right].$$

In case the fixed point equation has multiple solutions, the correct one is that solution for which the term

$$\frac{\eta}{\sigma^2} + \frac{\eta - \log \eta}{2\beta} - \sqrt{\frac{\eta}{2\pi\sigma^2}} \int_{\mathbb{R}} \log \left[ \cosh \left( \frac{\eta}{\sigma^2} x \right) \right] \exp \left( -\frac{\eta(x - 1)^2}{2\sigma^2} \right) dx$$

is smallest (Tanaka 2002).
For large SNR, i.e. \( \sigma^2 \to 0 \), we have \( \eta \to 1 \).
The asymptotic multiuser efficiency is given by

$$\eta^* = \lim_{\sigma^2 \to 0} \eta = \min_{x \in \{\pm 1, 0\}^K \setminus \{0\}} \frac{x^\dagger H^\dagger H x}{N}$$

(Verdú 1986).
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(Verdú 1986).

**Theorem:**
The optimum asymptotic multiuser efficiency converges to 1 almost surely as \( N, K \to \infty \), but

\[ \beta = \frac{K}{N} < \infty \]

fixed (Tse & Verdú 2000).
Asymptotic Multiuser Efficiency

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What about \( \beta \to \infty \)?
**Definition**

An $N \times K$ matrix $H$ is called uniquely detecting for an alphabet $B$, iff

$$Hb_1 = Hb_2 \Rightarrow b_1 = b_2$$

for all $b_1 \in B^K \ni b_2$. 

2. Results from Literature

**Uniquely Detecting Matrices**
2. Results from Literature

**Logarithmically Infinite Overload**

*Theorem:*
Let $N, K \to \infty$, but

$$\zeta = \frac{K}{N \log_3 K}$$

fixed.

Then, there does not exist any $H \in \{\pm 1\}^{N \times K}$ such that it is uniquely detecting for binary data, iff

$$\zeta > \log_3 4 \approx 0.79$$

(Lindström 1966).
Theorem:
Let $N, K \to \infty$, but

$$
\zeta = \frac{K}{N \log_3 K}
$$

fixed. Then, there does not exist any $H \in \{\pm 1\}^{N \times K}$ such that the optimum asymptotic multiuser efficiency is greater than zero, if

$$
\zeta > \log_3 4 \approx 0.79
$$

(Lindström 1966).
**Theorem:**

Let $N, K \to \infty$, but

$$\zeta = \frac{K}{N \log_3 K}$$

fixed. Let $H \in \{\pm 1\}^{N \times K}$.

Then, the probability that $H$ is uniquely detecting for binary data is 1, if

$$\zeta < \frac{1}{2}$$

(Erdős & Rényi 1963).
**Theorem:**

Let $N, K \to \infty$, but

$$\zeta = \frac{K}{N \log_3 K}$$

fixed. Let $H \in \{\pm 1\}^{N \times K}$.

Then, the optimum asymptotic multiuser efficiency converges to a nonzero value almost surely, if

$$\zeta < \frac{1}{2}$$

(Erdős & Rényi 1963).
2. Results from Literature

Summary of Known Results

- Lindström
- η
- ζ
- 0.79
- 0.5
- 0

- Tse-Verdú

- Erdös-Rényi

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**Theorem:**
Let $N, K \to \infty$, but

$$\zeta = \frac{K}{N \log_3 K}$$

fixed. Let $H \in \{\pm 1\}^{N \times K}$. Then, the optimum asymptotic multiuser efficiency converges to 1 almost surely if

$$\zeta < \frac{3}{8}$$

(Sedaghat, RRM, Marvasti 2013).
3. New Results

**Generalization of the Binary Existence Result**

**Theorem:**
Let $N, K \to \infty$, but

$$\zeta = \frac{K}{N \log_3 K}$$

fixed. Let $H \in \{\pm 1\}^{N \times K}$.

Then, the optimum asymptotic multiuser efficiency converges to value which is greater or equal to

$$\min\{1, 4 - 8\zeta\}$$

(Sedaghat, RRM, Marvasti 2013).
3. New Results

**Summary of Results**

- **Lindström**
  - \( \eta \)
  - \( \zeta \)
  - 0.79
  - 0.5
  - 0
  - 1

- **Tse-Verdú**
  - \( \eta \)
  - \( \zeta \)
  - 0.79
  - 0.5
  - 0

- **Erdös-Rényi**
  - \( \eta \)
  - \( \zeta \)
  - 0
  - 0.5
  - 0.79
Summary of Results

- Summary of Results

- 3. New Results
3. New Results

Minimum Distance

- By Erdős and Rényi, the minimum distance is non-zero almost surely as long as $\zeta < \frac{1}{2}$ (despite logarithmically infinite overload).

- By the new result, the minimum distance is almost surely at least as big as in an hypercube downscaled by a factor of $\beta$ as long as $\zeta < \frac{3}{8}$ (despite logarithmically infinite overload).
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The projection reduces the noise variance from $K$ to $N$. 
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The projection reduces the noise variance from $K$ to $N$.

In a hypercube the kissing number (number of nearest neighbors is linear in $K$.
After the projection it could eventually be exponential in $K$ degrading the asymptotic multiuser efficiency.
**Theorem:**

Let $N, K \to \infty$, but

$$
\zeta = \frac{K}{N \log_3 K}
$$

fixed. Let $H$ have Gaussian entries. Then, the optimum asymptotic multiuser efficiency converges to 1 almost surely if

$$
\zeta < \frac{1}{2}
$$

(Sedaghat, RRM, Marvasti 2013).
4. Proofs

**Sketch of Proof**

We define $\mathbf{R} = \mathbf{H}^\dagger \mathbf{H} / N$ and start from Verdú 1986

$$\Pr(\eta^* < 1) = \Pr \left( \min_{\mathbf{x} \in \{\pm 1, 0\}^K \setminus \{0\}} \mathbf{x}^\dagger \mathbf{R} \mathbf{x} < 1 \right).$$

The union bound gives

$$\Pr(\eta^* < 1) \leq \sum_{\mathbf{x} \in \{\pm 1, 0\}^K \setminus \{0\}} \Pr \left( \mathbf{x}^\dagger \mathbf{R} \mathbf{x} < 1 \right)$$

We split the summation into shells of equal Hamming weight (zero norm):

$$\sum_{\mathbf{x} \in \{\pm 1, 0\}^K \setminus \{0\}} (\cdot) = \sum_{w=1}^{K} \sum_{\mathbf{x} \in \{\pm 1, 0\}^K : \|\mathbf{x}\|_0 = w} (\cdot)$$
Lemma 1:
For any vector $x \in \{\pm 1, 0\}^K$ with odd weight,

$$x^\dagger Rx \geq 1.$$  

Proof:

$$x^\dagger Rx = \frac{1}{N} \sum_{i=1}^{N} \left( \sum_{j=1}^{K} H_{ij} x_j \right)^2 \geq 1$$

The squared bracket is a positive integer, since the sum contains an odd number of binary antipodal ($+1, -1$) terms.
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The squared bracket is a positive integer, since the sum contains an odd number of binary antipodal ($+1, -1$) terms.

*W. l. o. g., we can consider vectors with even weights.*
Lemma 2:
For any vector $x \in \{\pm 1, 0\}^K$ with even weight and

$$x^\dagger Rx < 1,$$

we have

$$\|Hx\|_0 < \frac{N}{4}$$

Proof:

$$x^\dagger Rx = \frac{1}{N} \sum_{i=1}^{N} \left( \sum_{j=1}^{K} H_{ij} x_j \right)^2$$

If the empirical average is less than 1, the number of terms which are 4 or more must be less than $\frac{N}{4}$. 
Lemma 2:
For any vector \( x \in \{ \pm 1, 0 \}^K \) with even weight and
\[
x^\dagger R x < 1,
\]
we have
\[
\| H x \|_0 < \frac{N}{4}
\]

Proof:
\[
x^\dagger R x = \frac{1}{N} \sum_{i=1}^{N} \left( \sum_{j=1}^{K} H_{ij} x_j \right)^2
\]

If the empirical average is less than 1, the number of terms which are 4 or more must be less than \( N/4 \).
**Lemma 3:**
For any vectors $x, y \in \{\pm 1, 0\}^K$ with

$$\|x\|_0 = \|y\|_0,$$

we have

$$\Pr \left( x^\dagger R x < 1 \right) = \Pr \left( y^\dagger R y < 1 \right)$$

**Proof:**

$$x^\dagger R x = \frac{1}{N} \sum_{i=1}^{N} \left( \sum_{j=1}^{K} H_{ij} x_j \right)^2$$

$$=: u_i$$
Lemma 3:
For any vectors $x, y \in \{\pm 1, 0\}^K$ with
\[ \|x\|_0 = \|y\|_0, \]
we have
\[ \Pr(x^\dagger Rx < 1) = \Pr(y^\dagger Ry < 1) \]

Proof:
\[
x^\dagger Rx = \frac{1}{N} \sum_{i=1}^{N} \left( \sum_{j=1}^{K} H_{ij} x_j \right)^2 =: u_i
\]

Characteristic function:
\[
\phi_{u_i}(t) = \prod_{j=1}^{K} \phi_{H_{ij} x_j}(t) = [\phi_{H_{11}}(t)]^\|x\|_0
\]
After splitting into subshells, we had:

\[
\Pr(\eta^* < 1) \leq \sum_{w=1}^{K} \sum_{x \in \{\pm 1, 0\}^K : \|x\|_0 = w} \Pr(x^\dagger R x < 1)
\]
Back to the Main Proof

After splitting into subshells, we had:

\[
\Pr(\eta^* < 1) \leq \sum_{w=1}^{K} \sum_{x \in \{\pm 1, 0\}^K : ||x||_0 = w} \Pr (x^\dagger Rx < 1)
\]

With the three lemmas, we get

\[
\Pr(\eta^* < 1) \leq \sum_{w=1}^{K/2} \binom{K}{2w} 2^{2w} \Pr \left( ||Hx||_0 < \frac{N}{4} \right)
\]
After splitting into subshells, we had:

\[
\Pr(\eta^* < 1) \leq \sum_{w=1}^{K} \sum_{x \in \{\pm 1, 0\}^K : \|x\|_0 = w} \Pr(x^\dagger R x < 1)
\]

With the three lemmas, we get

\[
\Pr(\eta^* < 1) \leq \frac{K}{2} \sum_{w=1}^{K/2} \binom{K}{2w} 2^{2w} \Pr\left(\|H x\|_0 < \frac{N}{4}\right)
\]

\[
= \frac{K}{2} \sum_{w=1}^{K/2} \binom{K}{2w} 2^{2w} \sum_{i=0}^{N/4-1} \binom{N}{i} p_w^{N-i} (1 - p_w)^i
\]

with

\[
p_w = \Pr\left(\sum_{j=1}^{2w} H_{1j} = 0\right) = \binom{2w}{w} 2^{-2w}.
\]

A few more bounds lead to the desired result.
5. References

**References**


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