Large Deviations of the Smallest Eigenvalue of Wishart-Laguerre Ensemble

Eytan Katzav, Isaac Pérez Castillo

Disordered Systems Group
Department of Mathematics, King’s College London

VI Brunel Workshop on Random Matrix Theory
17-18 December 2010
Introduction

- Random matrices, the Wishart ensemble
- The smallest eigenvalue
- Known results: exact approaches for small $N$, Tracy-Widom for large $N$

Coulomb Gas Approach

- Exact Mapping
- Continuing theory
- Saddle-point evaluation (large $N$)
- Results

Future

- Entropy corrections, etc.
Wishart-Laguerre Ensemble

- Experiment: where $N$ quantities are measured $M$ times

$$X^T = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1M} \\ \vdots & & & \vdots \\ x_{N1} & x_{N2} & \cdots & x_{NM} \end{pmatrix}$$

$N$ quantities, $W = X^T X$

- $M$ repetitions

- The Wishart-Laguerre ensemble is the random-matrix version, i.e. $x_{ij} \sim N(0, 1)$

- Normally $N < M$ (better $N \ll M$)

- When $N = M$, square matrices

- When $N \sim N$, $M - N = \mathcal{O}(1)$, almost square matrices

- Sometimes $N > M$ (anti-Wishart ensemble)
Wishart-Laguerre Ensemble

- We consider Wishart ensemble (Biometrica, 1928)
- Distribution of the $M \times N$ matrix $X$ is Gaussian

$$P(X) \sim \exp \left[ -\frac{\beta}{2} \text{Tr} X^\dagger X \right]$$

$(\beta = 1, 2, 4$ Dyson index$)$
- Wishart matrix $W = X^T X$
- In recent years, increased interest in so-called generalised $\beta$-ensembles (Dumitriu, 2003)
Density of Eigenvalues

- From distribution of Wishart matrices $\Rightarrow$ joint distribution of $N$ eigenvalues

$$\varrho_N(\lambda_1, \ldots, \lambda_N) = \frac{1}{Z_0} e^{-\frac{\beta}{2} \sum_{i=1}^N \lambda_i} \prod_{i=1}^N \lambda_i^{\frac{\beta}{2} (1+M-N)-1} \prod_{j<k} |\lambda_j - \lambda_k|^\beta$$

- First interesting object of study is the spectral density

$$\rho_N(\lambda) = \frac{1}{N} \sum_{i=1}^N \delta(\lambda - \lambda_i)$$
For large $N$, $\rho_N(\lambda) = (1/N)f(\lambda/N)$ follows the Marčenko-Pastur law (1967)

\[
f(x) = \frac{\sqrt{(\zeta_+ - x)(x - \zeta_-)}}{2\pi x} \mathbb{1}_{x \in [\zeta_-, \zeta_+]},
\]

\[
\zeta_- = \left(\frac{1}{\sqrt{c}} - 1\right)^2, \text{ (hard edge)}
\]

\[
\zeta_+ = \left(\frac{1}{\sqrt{c}} + 1\right)^2, \text{ (soft edge)}
\]

\[
c = \frac{N}{M}
\]

$c = 1/10$, $\zeta_- = 4.678$, $\zeta_+ = 17.324$
The Smallest Eigenvalue

Mathematics

- Invertibility of Wishart matrix is controlled by $\lambda_{\min}$
- Compressive sensing: fluctuations of $\lambda_{\min}$ set bounds on $\#$ of measurements to fully recover a sparse signal

Statistics

- Statistical tests based on $W^{-1}$ (e.g. Hotelling’s $T$-square test)

Physics

- Quantum information - measure of entanglement
The Smallest Eigenvalue

Exact expressions for finite $N$ and $M$ using various techniques, e.g.

- Edelman’s approach (1991)

$$\rho_{M,N}^{(\text{min})} (\lambda) = C_{M,N} \lambda^{(M-N-1)/2} e^{-\lambda N/2} g_{M,N}(\lambda)$$

with $g_{M,N}(\lambda)$ polynomials (different expressions for $M - N$ even or odd).

These expressions (and similar ones) difficult to evaluate for large sizes.

For large $N$, information on the typical fluctuations of the smallest eigenvalue ($c < 1$): Tracy-Widom distribution (Feldheim & Sodin, 2010)

$$\lambda_{\text{min}} = \zeta_- - \zeta_-^{2/3} c^{1/6} N^{-2/3} \chi_\beta, \quad \chi_\beta \sim TW_\beta$$
Our Goal

Study large fluctuations of the smallest eigenvalue

- simple expressions for rate functions for large deviations.

\[
P_N^{(\text{min})}(t) \sim e^{-\beta N^2 \Phi_+^{(\text{min})} \left( \frac{t - N\zeta_-}{N} \right)}, \quad N\zeta_- \leq t < \infty
\]

\[
P_N^{(\text{min})}(t) \sim e^{-\beta N\Phi_-^{(\text{min})} \left( \frac{N\zeta_- - t}{N} \right)}, \quad 0 \leq t \leq N\zeta_-
\]

- valid for any \( \beta \) and any \( c \)
Coulomb Gas approach

- From joint distribution of eigenvalues
  \[ \rho_N(\lambda) = \frac{1}{Z_0} e^{-\beta \sum_{i=1}^{N} \lambda_i \prod_{i=1}^{N} \lambda_i^{\frac{\beta}{2}(1+M-N)-1} \prod_{j<k} |\lambda_j - \lambda_k|^\beta} \]

- Coulomb Gas: eigenvalues as a system of charged particles in a 2D world (logarithmic potential), constrained to the real line and external linear-log potential
  \[ \rho_N(\lambda) = \frac{e^{-\beta F(\lambda)/2}}{Z_0} \]

with

\[
F(\lambda) = \sum_{i=1}^{N} \lambda_i - \left(1 + M - N - \frac{2}{\beta}\right) \sum_{i=1}^{N} \log \lambda_i - \sum_{i \neq j} \log |\lambda_i - \lambda_j|
\]
Coulomb Gas approach

- Quantity to calculate:

\[ P_N^{(\min)}(t) \equiv \text{Prob}(\lambda_{\min} \geq t) = \int_t^\infty d\lambda \rho_N^{(\min)}(\lambda) = \frac{Z(t)}{Z_0} \]

with

\[ Z(t) = \int_t^\infty \cdots \int_t^\infty e^{-\frac{\beta}{2} F(\lambda)} d\lambda_1 \cdots d\lambda_N \]

and \( Z_0 = Z(t = 0) \).

- Coulomb gas with hard wall at \( t \).
Continuum approach

- Introduce the density of particles
  
  \[ \rho(\lambda) = \frac{1}{N} \sum_{i=1}^{N} \delta(\lambda - \lambda_i) \]

- Change variables: \((\lambda_1, \ldots, \lambda_N) \rightarrow \rho(\lambda)\)
  
  \[
  Z(t) = \int_t^\infty d\lambda e^{-\frac{\beta}{2} F(\lambda)} = \int D\rho \ e^{-\frac{\beta}{2} F[\rho(\lambda)]} \int_t^\infty d\lambda \delta(F) \left[ \rho(\lambda) - \frac{1}{N} \sum_{i=1}^{N} \delta(\lambda - \lambda_i) \right]
  \]

- Rescaling: \(\rho(\lambda) = f(\lambda/N)/N\)
To obtain

\[ Z(t) = \int Df \ e^{-\frac{\beta N}{2} S[f(x)]} \]

with

\[ S[f(x)] = \int_\zeta^\infty dx f(x)x - \left( \alpha + \frac{\beta - 2}{\beta N} \right) \int_\zeta^\infty dx f(x) \log x \]

\[ - \int_\zeta^\infty dx dy f(x)f(y) \log |x - y| \]

\[ + \frac{2}{\beta N} \int_\zeta^\infty dx f(x) \log f(x) + C_1 \left( \int_\zeta^\infty dx f(x) - 1 \right) \]

with \( \alpha = \frac{(1 - c)}{c} \).

- No Dyson correction in the entropic term
Analytics - Saddle-point method

Stationary approximation (neglecting $1/N$ terms):

$$0 = \frac{\delta S[f(x)]}{\delta f(x)}$$

and differentiation with respect $x$ we obtain

$$\frac{1}{2} - \frac{\alpha}{2x} = P \int_{\zeta}^{\infty} dx' \frac{f_*(x')}{x - x'} , \quad \zeta \leq x < \infty$$

Tricomi equation
Analytics: Finite Interval Hilbert Transformation

Solution (Mathematical solution + normalisation + positivity):

\[ f_*(x) = \frac{\sqrt{U - x}}{\sqrt{x - \zeta}} \left( \frac{x - \alpha \sqrt{\zeta/U}}{x} \right)^{\mathbb{1}_{x \in [\zeta, U]}} , \quad \zeta \geq \zeta_- \]

with \( U \equiv U(c, \zeta) = w^2(c, \zeta) \) with

\[ w(c, \zeta) = \frac{2p}{3\rho^{1/3}} \cos \left( \frac{\theta + 2\pi}{3} \right) , \quad p = -[\zeta + 2(2 + \alpha)] \]

\[ q = 2\alpha \sqrt{\zeta} , \quad \rho = \sqrt{-\frac{p^3}{27}} , \quad \theta = \arctan \left( \frac{2\sqrt{B}}{q} \right) \]

\[ B = -\left( \frac{p^3}{27} + \frac{q^2}{4} \right) \]

For \( \zeta \leq \zeta_- \) we have \( U(c, \zeta_-) = \zeta_+ \left( \sqrt{\zeta_- \zeta_+} = \alpha \right) \)

Eytan Katzav, Isaac Pérez Castillo

Disordered Systems Group Department of Mathematics, King's College London

Large Deviations of the Smallest Eigenvalue of Wishart-Laguerre Ensemble
An intuitive representation
Large Deviations of the Smallest Eigenvalue of Wishart-Laguerre Ensemble
An intuitive representation

Eytan Katzav, Isaac Pérez Castillo
Disordered Systems Group Department of Mathematics, King’s College London

Large Deviations of the Smallest Eigenvalue of Wishart-Laguerre Ensemble
Large Deviations of the Smallest Eigenvalue of Wishart-Laguerre Ensemble
Large Deviations of the Smallest Eigenvalue of Wishart-Laguerre Ensemble
An intuitive representation

Eytan Katzav, Isaac Pérez Castillo
Disordered Systems Group Department of Mathematics, King's College London
Large Deviations of the Smallest Eigenvalue of Wishart-Laguerre Ensemble
Back to the distribution for $\lambda_{\text{textmin}}$

Recall $Z(t) = \int Df \, e^{-\frac{\beta}{2}N^2S[f(x)]}$

$$P^{(\text{min})}_N(t) = \frac{Z(t)}{Z_0} \sim e^{-\beta N^2 \Phi^{(\text{min})}_+ \left( \frac{t-N\zeta_-}{N} \right)}, \quad N\zeta_- \leq t < \infty$$

Right rate function

$$\Phi^{(\text{min})}_+(x) = \frac{1}{2} \left[ S(x + \zeta_-) - S(\zeta_-) \right], \quad 0 \leq x < \infty$$

with

$$S(\zeta) = \frac{\zeta + U}{2} - \frac{\Delta^2}{32} - \log \left( \frac{\Delta}{4} \right) + \frac{\alpha}{4} \left( \sqrt{U} - \sqrt{\zeta} \right)^2$$

$$+ \frac{\alpha^2}{4} \log(\zeta U) - \alpha(\alpha + 2) \log \left( \frac{\sqrt{\zeta} + \sqrt{U}}{2} \right)$$

with $\Delta = U - \zeta$
Large deviations to the left of $\lambda_{\text{min}}$

- Coulomb Gas approach (as presented) not able to capture fluctuations to the left of $\lambda_{\text{min}}$
- Reason: we only consider leading terms $O(N^2)$, which capture bulk properties
Large deviations to the left of $\lambda_{\text{min}}$

- Energetic Argument (Majumdar & Vergassola)
- Expression the free energy $F(\lambda)$
- Energetic cost of moving the smallest eigenvalue to the left $t \ll \zeta N$ (this does not require a global rearrangement of the bulk)

$$
\Delta E(t) = F(t, \lambda_2, \ldots, \lambda_N) - F(\zeta N, \lambda_2, \ldots, \lambda_N)
= t - \alpha N \log(t) - 2 \sum_k \log |t - \lambda_k| + C
= t - \alpha N \log(t) - 2N \int d\lambda \rho_{\text{MP}}(\lambda) \log |t - \lambda| + C
$$

$C$ so that $\Delta E(t = \zeta N) = 0.$
Large deviations to the left of $\lambda_{\text{min}}$

- Obtain

$$P_N^{(\text{min})}(t) \sim e^{-\beta N \Phi_{-}^{(\text{min})}\left(\frac{N\zeta_{-} - t}{N}\right)}, \quad 0 \leq t \leq N\zeta_{-}$$

- Left rate function

$$\Phi_{-}^{(\text{min})}(x) = -\frac{\alpha}{2} \log \left(1 - \frac{x}{\zeta_{-}}\right) - \frac{1}{2} \sqrt{x(x + \Delta_{-})}$$

$$+ 2 \log \left(\frac{\sqrt{x + \Delta_{-}} - \sqrt{x}}{\sqrt{\Delta_{-}}}\right)$$

$$+ \alpha \log \left(1 + 2 \frac{\sqrt{x(x + \Delta_{-})} - x}{\Delta_{-} \sqrt{\zeta_{-}}}\right), \quad 0 \leq x \leq \zeta_{-}$$

with $\Delta_{-} = \zeta_{+} - \zeta_{-} = 4\sqrt{1 + \alpha}$. 
Large deviations - Numerics

\( N = 11, \ M = 110. \) Comparison with Edelman’s (91) for \( \beta = 1 \)

\[
-N \Phi_+ \left( \frac{t - N \zeta}{N} \right)
\]

\[
N^2 \Phi_+ \left( \frac{t - N \zeta}{N} \right)
\]
Comparison with Tracy-Widom

\[ P_N^{(\text{min})}(t) \equiv \lim_{N \to \infty} \mathbb{P}_{\beta,N} \left( (\lambda_{\text{min}} - z_N^{(\beta)})/s_N^{(\beta)} \leq t \right) \]

To compare with Tracy-Widom, expand rate functions:

\[
\Phi_-^{(\text{min})}(x) \sim_{x \to 0} \frac{2}{3\zeta_- c^{1/4}} x^{3/2}, \quad \Phi_+^{(\text{min})}(x) \sim_{x \to 0} \frac{1}{24\zeta_-^2 \sqrt{c}} x^3
\]

Then

\[
P_N^{(\text{min})}(t) \sim \begin{cases} 
\exp \left( -\frac{2\beta}{3} \chi^{3/2}(t) \right) , & 0 \leq t \leq \zeta_- N \\ 
\exp \left( -\frac{\beta}{24} |\chi(t)|^3 \right) , & t > \zeta_- N
\end{cases}
\]

with \( \chi(t) = -\frac{N\zeta_- - t}{N^{1/3} \zeta_-^{2/3} c^{1/6}} \)
Almost Square Matrices

- \( M = N + a, \alpha = a/N \), \( a \rightarrow a(\beta) = a + (\beta - 2)/\beta \)
- Look at the behaviour for \( z = Nt \)

\[
P_N^{(\min)}(z) \sim \begin{cases} 
\exp\left(-\beta a \psi_-(\min)\left(\frac{4z}{a^2}\right)\right), & z \in [0, a^2/4] \\
\exp\left(-\beta a^2 \psi_+(\min)\left(\frac{4z}{a^2}\right)\right), & z \in [a^2/4, \infty)
\end{cases}
\]

with

\[
\psi_+(\min)(x) = \frac{1}{8} \left(x - 4\sqrt{x} + 3 + \ln x\right),
\]

\[
\psi_-(\min)(x) = \ln \frac{1 + \sqrt{1-x}}{\sqrt{x}} - \sqrt{1-x}
\]

- \( a(\beta) = 0 \) (\( a = 1, \beta = 1 \) or \( a = 0, \beta = 2 \))

\[
P_N^{(\min)}(z) = e^{-\beta z/2}
\]
Almost Square Matrices

Comparison with Edelman’s exact result for $\beta = 1$ ($N = 200$, $a=5$)
Subleading contributions

- Entropic contribution: Saddle-point equation

\[
\frac{1}{2} (x - \alpha \log x) + \frac{1}{\beta N} \log f(x) + D = \int_\zeta^\infty dy f(y) \log |x - y|
\]

Support of \( f(x) \) is not compact ⇒ fluctuations to the left of \( \lambda_{\text{min}} \)

- Non-linear integral equation (Hammerstein type)

- Standard perturbation is hopeless

- Non-standard perturbation (boundary layer theory ?) as difficult as the original equation

Eytan Katzav, Isaac Pérez Castillo
Disordered Systems Group Department of Mathematics, King’s College London

Large Deviations of the Smallest Eigenvalue of Wishart-Laguerre Ensemble
Two options:

- simplest analytical approach: \( y \in \mathcal{R}_{\text{interior}}, \ x \in \mathcal{R}_{\text{exterior}} \),
  \[
  V(x) = \frac{1}{2} (x - \alpha \log x)
  \]
  \[
  f(x) \sim \exp \left[ -\beta N \left( V(x) - \int_{y \in \mathcal{R}_{\text{interior}}} dy \ f_{\text{MP}}(y) \log |x - y| \right) \right]
  \]
  (instanton contribution as in Fyodorov 2004)
Subleading contributions

- Numerical solution (Abdou & Ismail 2002)