Chaos in fermionic many-body systems and the metal-insulator transition

Hans A. Weidenmüller
MPI für Kernphysik
Heidelberg, Germany

In collaboration with T. Papenbrock (Oak Ridge National Laboratory and University of Tennessee), Z. Pluhar (Prague), and J. Tithof (University of Tennessee)
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1. Motivation. Metal-Insulator Transition

Finite many-body systems (atoms, molecules, nuclei) often display spectral fluctuation properties that agree with those of GOE. But in GOE all states coupled by independent matrix elements while such systems governed by few-body forces. Question: Does a k-body interaction generically produce chaos? (k = 2,3,…)

Difficulty: With increasing dimension Hamiltonian matrix for k-body interaction becomes infinitely sparse (fraction of nonzero nondiagonal elements goes to zero). Not analytically accessible. Numerical simulations very far from sparse limit.

Most extensive investigation combines perturbative estimates and numerical simulation and k=1 and k=2 interactions. Determines critical strength of k = 2 interaction for transition to chaos. Implies that k=2 interaction produces chaos. But far from sparse limit.

Different approach: Take matrix representation of random k-body interaction. Question: Which properties of resulting random-matrix ensemble affect spectral shape and/or spectral fluctuations?

Guide: Metal-Insulator Transition (Varga).

**Anderson model (1958)**

- **Hamiltonian:**
  \[ \mathcal{H} = \sum_{n} \varepsilon_{n} c_{n}^{\dagger} c_{n} + \sum_{n,m} V_{nm} c_{n}^{\dagger} c_{m} \]

- **Energies** \( \varepsilon \) are **uncorrelated**, random numbers from uniform (bimodal, Gaussian, Cauchy, etc.) distribution \( \Rightarrow W \)

- **Nearest-neighbor hopping** \( \Rightarrow V \)

![Diagram showing Anderson model](image)
Simulation in terms of power-law band random matrix. For $a > 1$, spectrum is Poissonian, for $a = 1$, it is critical, for $a < 1$, it is GOE-like.

Properties of the power-law random band matrix at the critical point \((a = 1, b = 1)\).

Left: Nearest-neighbor spacing distribution \(P(s)\) versus \(s\). Strong deviation from Wigner distribution (dashed line). But only qualitative agreement with \(P(s) = 4s \exp(-2s)\).

Right: Number variance \(\Sigma^{(2)}\) versus interval length \(L\). For small \(L\) close to Wigner-Dyson, for large \(L\) close to linear. I. Varga, D. Braun, Phys. Rev. B 61 (2000) R11859.

Expand eigenfunctions \(\Psi_\mu\) in fixed basis,
\[\Psi_\mu = \sum_{\nu} a_{\mu \nu} |\nu\rangle.\]
Define inverse participation ratio \(P_{2}(\mu) = \sum_{\nu} |a_{\mu \nu}|^4\). Expect distribution of \(P_{2}(\mu)\) to be proportional to \(N^{-D}\) with \(D\) between zero and one.
Show distribution of \(\ln P_{2}\) for several values of \(N\).

Idea: Construct random-matrix ensemble that is more sparse than for any \(k\)-body interaction and has same spectral properties as critical ensemble.
2. Embedded k-Body Random Ensemble

Consider \( l \) degenerate single-particle states and \( m \) Fermions with random k-body interaction.

\[
H = \sum_{i_1 \ldots i_k; j_1 \ldots j_k} v_{i_1 \ldots i_k; j_1 \ldots j_k} a_{i_1}^\dagger \times \ldots \times a_{i_k}^\dagger a_{j_k} \times \ldots \times a_{j_1}
\]

The k-body interaction matrix elements are Gaussian-distributed real random variables with zero mean values and common second moment. Model avoids complexity of angular-momentum coupling. Consider \( l, m \gg 1 \). Is spectral statistic of \( \text{EGUE}(k) \) GUE-like?


Hilbert space spanned by Slater determinants has dimension \( N = \binom{l}{m} \approx l^m \). Matrix representation \( H_{\mu \nu} \) with \( \mu, \nu = 1, \ldots N \). Number of nonzero matrix elements in every row and column is

\[
\sum_{p=1}^{k} \binom{m}{p} \binom{l-m}{p}
\]

Fraction \( f \) of these elements \( \to 0 \) for \( N \to \infty \). Extremely sparse.

In that limit, spectral statistics not known. Numerical simulation suggests GOE but typically for \( l=12, m=6, N=924, k=2 \) we have \( f=0.28 \) not sparse!
3. Scaffolding Ensemble

Question: On which side of metal-insulator transition is EGOE(k)?

Idea: Construct random-matrix ensemble that is even more sparse than EGOE(k) and that is critical. Then EGOE(k) should have GOE statistics. Guide: Power-law random band matrix.

“Scaffolding Matrix” A:

A has dimension $N = 2^n$, with $n = 1, 2, \ldots$. Number of non-vanishing non-diagonal elements in every row and every column is $n = \ln N / \ln 2$. Thus for every $k$, A is more sparse than EGOE(k). With increasing distance $D$ from main diagonal, density of non-diagonal elements falls off like $1/D$. 
With help of $A$, define scaffolding ensemble $\text{ScE}$. Matrices have dimension $N = 2^n$. Matrix elements are real Gaussian random variables with zero mean and second moments

$$(1 + \delta_{\mu \nu}) \left< H_{\mu \nu}^{\text{sc}} H_{\rho \sigma}^{\text{sc}} \right> = (\delta_{\mu \sigma} \delta_{\nu \rho} + \delta_{\mu \rho} \delta_{\nu \sigma}) \left[ \alpha \delta_{\mu \nu} + A_{\mu \nu} \right].$$

Choose $\alpha >> 1$. Spectral properties of $\text{ScE}$:

(i) For $n >> \alpha$, average spectrum has Gaussian shape (Pastur eq.)

(ii) $\text{ScE}$ is much more sparse than $\text{EGOE}(k)$ for any $k$: For $n=13$, we have $f < 2 \times 10^{-3}$! So use numerical simulation. Test fluctuation measures of critical ensemble.
The diagram shows the probability distribution function $P(s)$ for different values of $n=10$ and $\alpha=3, 10, 30, 100$. The distributions are labeled as 'sc, n=10, $\alpha=3$', 'sc, n=10, $\alpha=10$', 'sc, n=10, $\alpha=30$', and 'sc, n=10, $\alpha=100$'. Additionally, there are special cases for Wigner and Poisson distributions.
Conclusion: For $\alpha >> 1$ and large $N$, ScE has critical statistic. That fact and the fact that EGOE(k) is much less sparse than ScE suggest that for all $k$, EGOE(k) obeys Wigner-Dyson statistic.

This argument is qualitative only. Quantitative argument: Levitov’s criterion.
4. Levitov’s Criterion


Criterion for the spectral properties of a dense random matrix.

$H_{\mu \nu}$ is dense (i.e., not sparse) random matrix (dimension N) with large diagonal elements ($<|H_{\mu \mu}|^2>$ $\gg$ $<|H_{\mu \nu}|^2>$ for $\mu \neq \nu$). Values of $<|H_{\mu \nu}|^2>$ decrease with increasing $|\mu - \nu|$. Renormalization group approach shows that spectrum of $H$ is Poissonian (GOE-like) if $S = \sum_\nu <|H_{\mu \nu}|^2>^{1/2} / <|H_{\mu \mu}|^2>^{1/2}$ converges (diverges) as $N \to \infty$. Critical ensemble: $S \propto \ln N$.

Criterion cannot be extended analytically to sparse matrices. Does it apply to critical and scaffolding ensembles? Numerical test for ScE with $\alpha = 17$, $n$ large. Result is: Scaffolding ensemble: $\sum_\nu <|H_{\mu \nu}|^2> \propto \ln N$ (critical).

Conclusion: Levitov’s criterion is applicable to sparse matrices.
5. EGOE(k) for k > 1

Assume that Levitov’s criterion does apply to EGOE(k). But careful! EGOE(k) differs from scaffolding ensemble in three ways.

(i) Number of non-diagonal elements in every row and column is \( \gg \ln N \approx m \ln l \): Much less sparse.

(ii) Number of k-body matrix elements contributing to given m-body matrix element bigger than one. Example diagonal: \( m^k \). That increases the variances. Example non-diagonal: At most \( m^{(k-1)} \). Thus, EGOE(k) satisfies assumptions of Levitov’s criterion.

(iii) m-body matrix elements in different rows and columns may contain same k-body matrix elements and be correlated.

Neglect correlations:
Sum of non-diagonal elements in every row and column diverges with \( N \to \infty \) much more strongly than \( \ln N \); Ensemble without correlations is GOE-like.
Influence of correlations:


(ii) Spectral fluctuations: Correlations only between matrix elements located in different rows and columns. Do not expect that these affect spectral fluctuation properties.

So expect that correlations deform average spectrum towards semicircle but do not affect spectral fluctuations.

Test this expectation for $k = 1$. 
6. EGOE(1)

EGUE(1) is special: One-body Hamiltonian can be diagonalized. Then m-body Hamiltonian diagonal, too. Eigenvalues are sums of one-body eigenvalues: Poisson statistics.
In m-body matrix representation: Same one-body matrix elements occur repeatedly in well-defined positions.

To test our idea, compare three ensembles.

(i) EGOE(1). Gaussian spectral shape, Poisson statistics.
EGOE(1) with randomized matrix elements

Scrambled EGOE(1)
7. Summary


ScE: a very sparse random-matrix ensemble with semicircular spectral shape and spectral statistics of critical ensemble. Levitov’s criterion works.

Levitov’s criterion applied to EGOE(k) without correlations gives Wigner-Dyson statistics.

For EGOE(1): Correlations affect spectral shape but not spectral fluctuations.

Convincing evidence that for k > 1, EGOE(k) has same spectral fluctuations as GOE. And we see why that is so.
$P(s)$

$n=13, \alpha=17$

- **GOE**
- **Poisson**
$\Sigma^2(L) = \begin{cases} \text{n=13, } \alpha=17 \\ \text{GOE} \\ \text{Poisson} \end{cases}$