Non-intersecting squared Bessel paths

ARNO KUIJLAARS

Department of Mathematics

Katholieke Universiteit Leuven

Belgium

joint work with

Andrei Martínez-Finkelshtein (Almeria, Spain)

Franck Wielonsky (Lille, France)

Determinantal point processes

- The structure behind eigenvalues of unitary random matrices appears in other situations as well
  - Non-intersecting paths
  - Certain tiling problems, stochastic growth problems
  - Representation theory of large groups
- They are determinantal point processes: a random point process so that correlation functions have determinantal form

$$\det\left[K(x_j, x_k)\right]_{j,k=1,\ldots,m}$$

  - K is the correlation kernel

Unitary random matrices

- **The eigenvalues of the unitary ensemble**

$$\frac{1}{Z_n} \exp(-\operatorname{Tr} V(M)) dM$$

defined on $n \times n$ Hermitian matrices M follow a determinantal point process with correlation kernel

$$K(x,y) = \sqrt{e^{-V(x)}}\sqrt{e^{-V(y)}} \sum_{j=0}^{n-1} p_j(x) p_j(y)$$

  - p_j **is the orthonormal polynomial of degree j with weight $e^{-V(x)}$**

Non-intersecting random paths

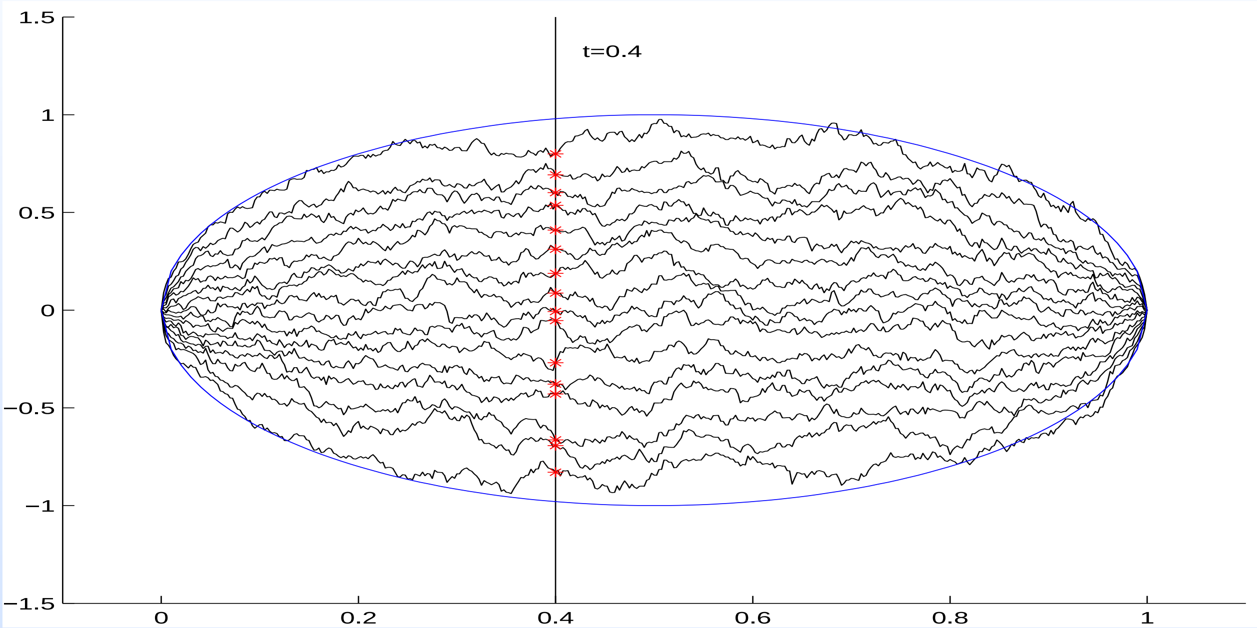
- **Given a 1-D diffusion process with transition probability density $p_t(x,y)$. Suppose n independent copies $X_1(t),\dots,X_n(t)$ conditioned to**
  - **start at time $t=0$ at n given points $a_1 < a_2 < \cdots < a_n$,**
  - **end at time $t=T$ at n given points $b_1 < b_2 < \cdots < b_n$,**
  - **not intersect in the full time interval $(0,T)$.**
- **Then the positions of the paths at given time $t \in (0,T)$ are a determinantal point process.** consequence of Karlin-McGregor (1959) theorem
- **The correlation kernel K takes the form**

$$K(x,y) = \sum_{j=1}^{n} \phi_j(x)\psi_j(y)$$

where biorthogonal functions ϕ_j and ψ_j arise from biorthogonalizing the transition probability density functions $p_t(a_j,x)$ and $p_{T-t}(x,b_j)$.

Confluent case as a model for GUE

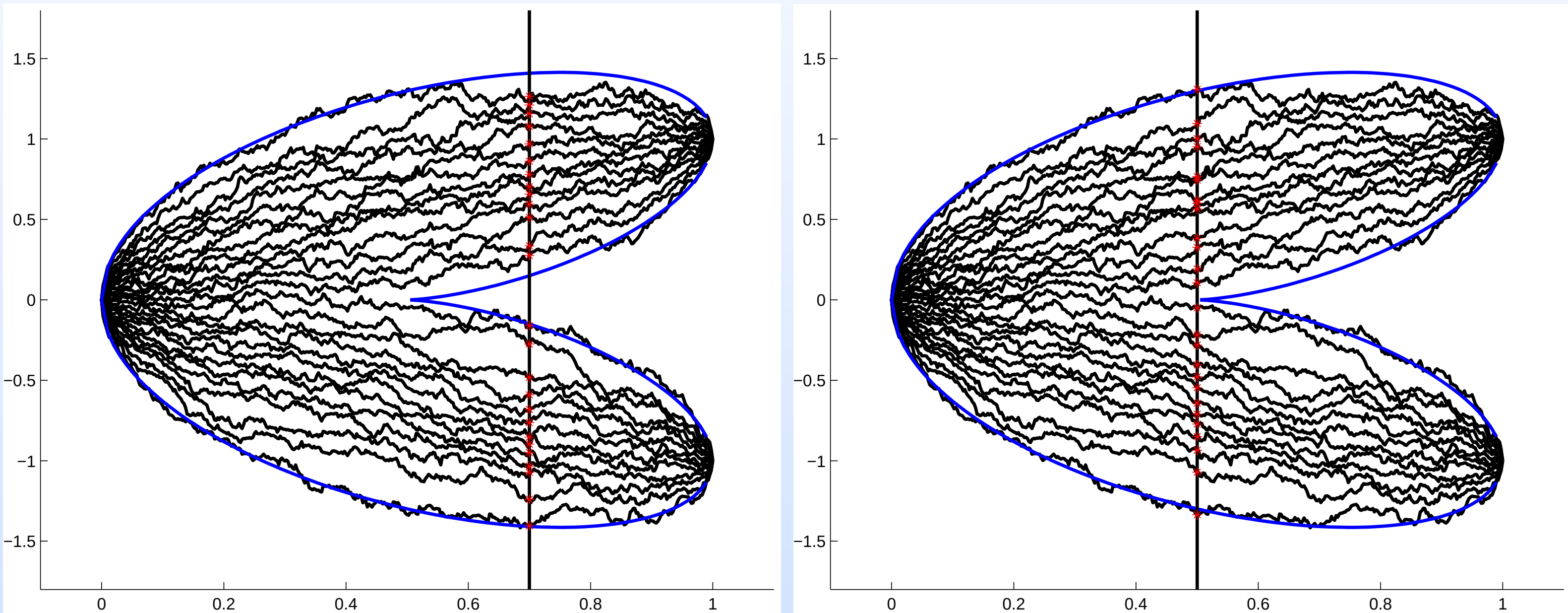
- The confluent case $a_j \to 0$, $b_j \to 0$, for Brownian motion leads to (a variation on) Dyson's Brownian motion



- The positions of the paths at any given time t have the same joint p.d.f. as the eigenvalues of an $n \times n$ GUE matrix.

GUE with external source

- Eigenvalues of GUE with **external source** having 2 distinct eigenvalues, have the same joint pdf as the positions of non-intersecting Brownian paths with two end positions



- Kernel has usual scaling limits (**sine kernel** and **Airy kernel**) from random matrix theory. At the cusp point a new family of scaling limits appear in a double scaling limit: **Pearcey kernels** Brézin-Hikami (1998), Bleher-Kuijlaars (2007)

Multiple Hermite polynomials

- We analyzed this using the connection with **multiple Hermite polynomials**.
  - Given two weight functions w_1 and w_2 and indices n_1, n_2, we look for a monic polynomial

$$P_{n_1,n_2}(x) = x^n + \cdots, \qquad n = n_1 + n_2$$

    such that

$$\int_{-\infty}^{\infty} P_{n_1,n_2}(x) x^k w_1(x)\, dx = 0, \qquad k = 0, \ldots, n_1 - 1,$$
$$\int_{-\infty}^{\infty} P_{n_1,n_2}(x) x^k w_2(x)\, dx = 0, \qquad k = 0, \ldots, n_2 - 1.$$

  - P_{n_1,n_2} is called a **multiple orthogonal polynomial** (MOP).
- Multiple Hermite polynomials correspond to case

$$w_1(x) = e^{-n(\frac{1}{2}x^2 - ax)}, \qquad w_2(x) = e^{-n(\frac{1}{2}x^2 + ax)}$$

Riemann-Hilbert problem for MOPs

- **MOPs (with two weights) satisfy a 3×3 matrix valued RH problem**

Van Assche-Geronimo-Kuijlaars (2001)

generalization of Fokas-Its-Kitaev (1992) RH problem for OPs

  - $Y : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}^{3\times 3}$ **is analytic**
  - $Y_+(x) = Y_-(x) \begin{pmatrix} 1 & w_1(x) & w_2(x) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ **for** $x \in \mathbb{R}$
  - $Y(z) \begin{pmatrix} z^{-n_1-n_2} & 0 & 0 \\ 0 & z^{n_1} & 0 \\ 0 & 0 & z^{n_2} \end{pmatrix} \to I_3$ **as** $z \to \infty$

- **The RH problem is solvable if and only if the MOP P_{n_1,n_2} exists and is unique.**

Correlation kernel

- The correlation kernel for non-intersecting Brownian paths with two endpoints can be expressed in terms of the solution Y of the RH problem

$$K(x,y) = \frac{1}{2\pi i(x-y)} \begin{pmatrix} 0 & w_1(y) & w_2(y) \end{pmatrix} Y_+(y)^{-1} Y_+(x) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

with weights $w_1(x) = e^{-n(x^2-ax)}$, $w_2(x) = e^{-n(x^2+ax)}$ where a depends on t

for two weights: Bleher-Kuijlaars (2004)

extension to more than two weights: Daems-Kuijlaars (2004)

- The RH problem is then analyzed with the steepest descent method for RH problems of Deift and Zhou.

Squared Bessel process

- **Squared Bessel process** is a diffusion process on $[0, \infty)$ depending on parameter $\alpha > -1$, with transition probabilities

$$p_t^{\alpha}(x,y) = \frac{1}{2t}\left(\frac{y}{x}\right)^{\alpha/2} e^{-(x+y)/(2t)} I_{\alpha}\left(\frac{\sqrt{xy}}{t}\right), \qquad x, y > 0,$$

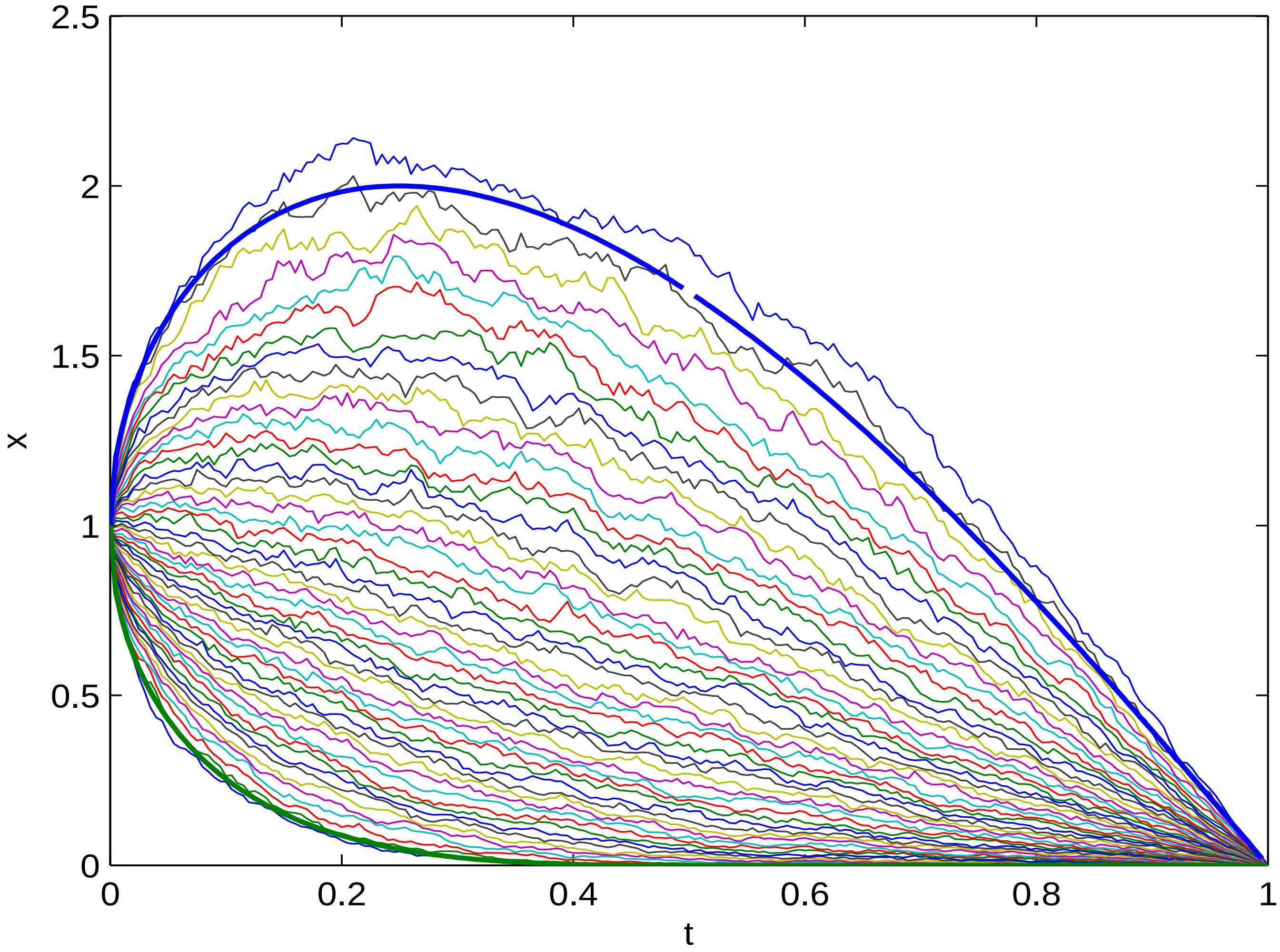
where I_α is the **modified Bessel function** of the first kind of order α

$$I_\alpha(z) = \sum_{k=0}^{\infty} \frac{(z/2)^{2k+\alpha}}{k!\Gamma(k+\alpha+1)}$$

- If $d = 2(\alpha+1)$ is an integer, then the squared Bessel process is the square of the distance to the origin of a d-dimensional Brownian motion.
  - Non-intersecting squared Bessel processes were considered by König-O'Connell (2001) and Desrosiers-Forrester (arxiv 2006).
  - Related work by Katori-Tanemura and by Tracy-Widom (2007) on non-intersecting Brownian excursions.

Simulation of 50 non-intersecting paths

- **Confluent case:** $a_j \to a > 0$ **and** $b_j \to 0$**.**



Non-intersecting squared Bessel paths

- **PROPOSITION:**

  In the confluent limit $a_j \to a > 0$, $b_j \to 0$, the positions of n non-intersecting squared Bessel paths at time $t \in (0,1)$ are a MOP ensemble with weights

$$w_1(x) = x^{\alpha/2} e^{-\frac{Tx}{2t(T-t)}} I_\alpha\left(\frac{\sqrt{ax}}{t}\right)$$

$$w_2(x) = x^{(\alpha+1)/2} e^{-\frac{Tx}{2t(T-t)}} I_{\alpha+1}\left(\frac{\sqrt{ax}}{t}\right)$$

  and multi-indices $n_1 = \lceil n/2 \rceil$, $n_2 = \lfloor n/2 \rfloor$

  - Proof depends on differential relations for modified Bessel functions

RH problem

- **COROLLARY: The correlation kernel K_n is expressed in terms of the solution Y of a 3×3 matrix valued RH problem**

$$K_n(x,y) = \frac{1}{2\pi i(x-y)} \begin{pmatrix} 0 & w_1(y) & w_2(y) \end{pmatrix} Y_+(y)^{-1} Y_+(x) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

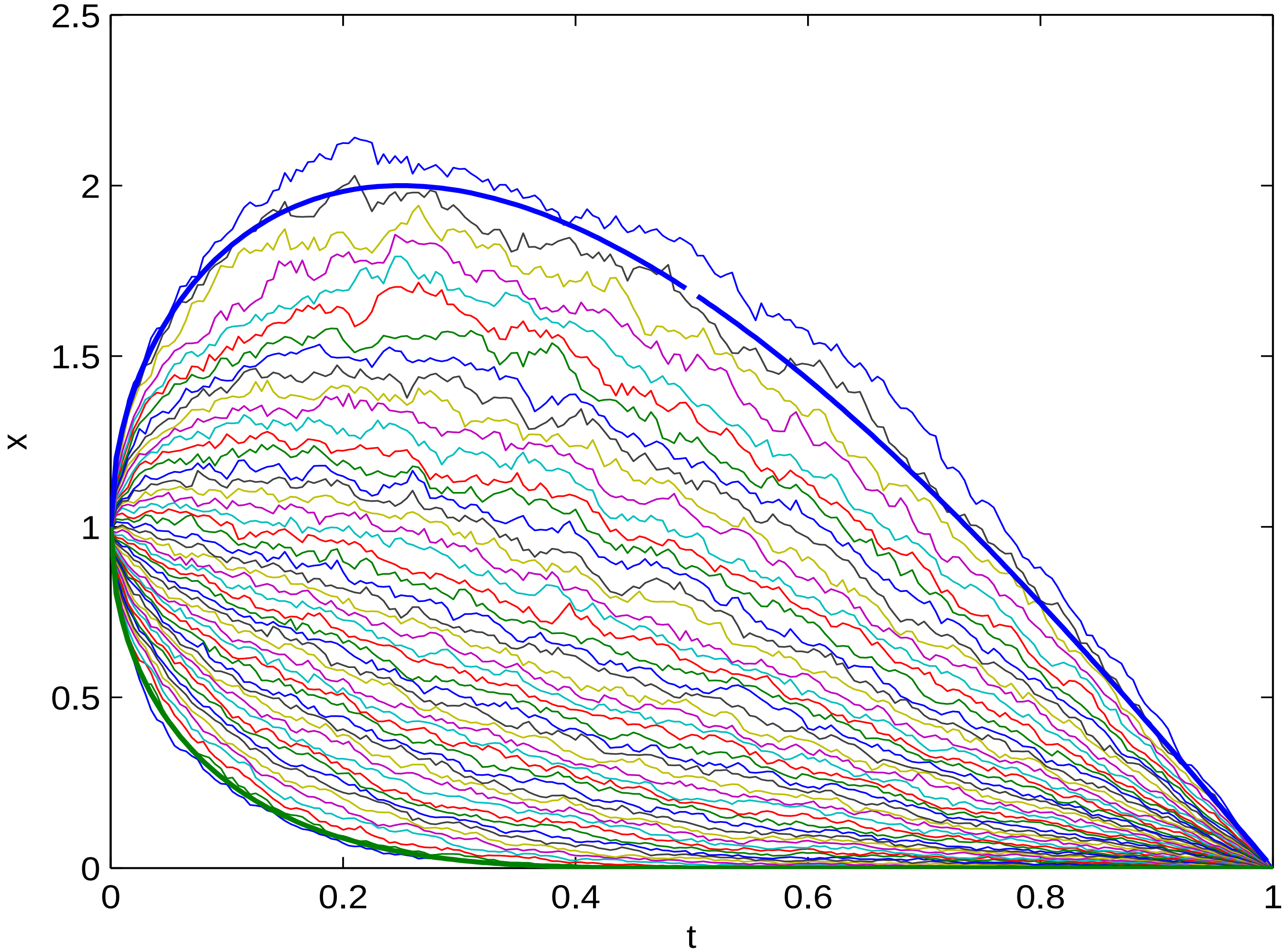
  - **Jump condition** $\quad Y_+(x) = Y_-(x) \begin{pmatrix} 1 & w_1(x) & w_2(x) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ **on** $\mathbb{R}$
  - **Asymptotic condition**

    $Y(z) = (I + O(1/z))\,\mathrm{diag}\begin{pmatrix} z^{n_1+n_2} & z^{-n_1} & z^{-n_2} \end{pmatrix}$

- **To obtain interesting limits as $n \to \infty$, we rescale time**

$$t \mapsto \frac{t}{2n}, \qquad T \mapsto \frac{1}{2n}, \qquad \text{so now } 0 < t < 1$$

Simulation of 50 non-intersecting paths



Global regime 1

THEOREM on global regime

- **As $n \to \infty$ the squared Bessel paths fill out a region in the tx-plane whose boundary satisfies the algebraic equation**

$$4ax^3 + x^2(t^2 - 20at(1-t) - 8a^2(1-t)^2) - 4x(1-t)(t - a(1-t))^3 = 0$$

- **For every $0 < t < 1$ the algebraic equation has three solutions**

$$x = 0, \quad x = p = p(t), \quad x = q = q(t) \qquad \text{with } p < q$$

- **There is a critical time** $t_{cr} = \frac{a}{a+1}$ **so that**
  - $0 < p < q$ **for** $0 < t < t_{cr}$
  - $p < 0 < q$ **for** $t_{cr} < t < 1$

Global regime 2

THEOREM on global regime (cont')

- **For every $t \in (0,1)$ the paths at time t accumulate on the interval**

$$\Delta_1 = [\max(0,p), q]$$

with a limiting mean density

$$\frac{d\mu_1(x)}{dx} = \frac{1}{\pi}\left|\mathrm{Im}\,\zeta(x)\right|, \quad x \in \Delta_1,$$

where $\zeta = \zeta(x)$ is a non-real solution of

$$x\zeta^3 - \frac{2x}{t(1-t)}\zeta^2 + \left(\frac{x}{t^2(1-t)^2} + \frac{1}{t(1-t)} - \frac{a}{t^2}\right)\zeta - \frac{1}{t^2(1-t)^2} = 0$$

THEOREM on local regime

- **The correlation kernel $K_n(x,y)$ has scaling limits as $n \to \infty$**
  - **Bulk scaling near a point $x^* \in (p,q)$:**

$$\frac{\sin \pi(x-y)}{\pi(x-y)}$$

  - **Soft edge scaling near a point $x^* = q$ or $x^* = p$ if $t < t_{cr}$:**

$$\frac{\mathrm{Ai}(x)\,\mathrm{Ai}'(y) - \mathrm{Ai}'(x)\,\mathrm{Ai}(y)}{x-y}.$$

  - **Hard edge scaling near $x^* = 0$ if $t > t_{cr}$:**

$$\frac{J_\alpha(\sqrt{x})\sqrt{y}J'_\alpha(\sqrt{y}) - \sqrt{x}J'_\alpha(\sqrt{x})J_\alpha(\sqrt{y})}{2(x-y)}$$

THEOREM at the critical time

- **At the critical time t_{cr} the density of paths grows like $\sim x^{-1/3}$ as $x \to 0+$.**
- **Double scaling limit at $x^* = 0$ and $t = t_{cr}$:**

$$\lim_{n\to\infty} \frac{1}{cn^{3/2}} K_n\left(\frac{x}{cn^{3/2}}, \frac{y}{cn^{3/2}}; t = t_{cr} + \frac{\tau}{c'n^3}\right) = ??$$

- **New limiting kernels involving solutions of third order ODEs**
- **In case $\alpha = 0$:**

$$\frac{1}{\pi(x-y)} \left[f(x)yg''(y) - xf'(x)g'(y) + xf''(x)g(y) + \frac{x}{y}f'(x)g(y) - \tau f(x)g(y)\right]$$

  - $f(x)$ **is a solution of** $xf''' + 2f'' - \tau f' - f = 0$
  - $g(y)$ **is a solution of** $yg''' + g'' - \tau g' + g = 0$

About the proof

- **Deift-Zhou steepest descent analysis of the RH problem with jump matrix**

$$\begin{pmatrix} 1 & x^{\alpha/2} e^{-\frac{nx}{t(1-t)}} I_\alpha\left(\frac{2n\sqrt{ax}}{t}\right) & x^{(\alpha+1)/2} e^{-\frac{nx}{t(1-t)}} I_{\alpha+1}\left(\frac{2n\sqrt{ax}}{t}\right) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

for $x > 0$

First transformation $Y \mapsto X$

- The first transformation uses the special differential properties of

$$z^{\alpha/2} I_\alpha(2\sqrt{z}), \qquad z^{(\alpha+1)/2} I_{\alpha+1}(2\sqrt{z}),$$

$$z^{\alpha/2} K_\alpha(2\sqrt{z}), \qquad z^{(\alpha+1)/2} K_{\alpha+1}(2\sqrt{z})$$

- It leads to RH problem with jump matrices

$$\begin{pmatrix} 1 & x^\alpha e^{-n\left(\frac{x}{t(1-t)} - \frac{2\sqrt{ax}}{t}\right)} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad x > 0,$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{2in\frac{2\sqrt{|ax|}}{t}} & 0 \\ 0 & |x|^\alpha & e^{-2in\frac{2\sqrt{|ax|}}{t}} \end{pmatrix}, \quad x < 0$$

Equilibrium problem

- **Minimize**

$$\iint \log \frac{1}{|x-y|} d\mu_1(x) d\mu_1(y) - \iint \log \frac{1}{|x-y|} d\mu_1(x) d\mu_2(y)$$
$$+ \iint \log \frac{1}{|x-y|} d\mu_2(x) d\mu_2(y) + \int \left(\frac{x}{t(1-t)} - \frac{2\sqrt{ax}}{t} \right) d\mu_1(x)$$

over pairs of measures (μ_1, μ_2) **such that**

- $\mathrm{supp}(\mu_1) \subset [0, \infty)$**,** $\int d\mu_1 = 1$**,**
- $\mathrm{supp}(\mu_2) \subset (-\infty, 0]$**,** $\int d\mu_2 = 1/2$**,**
- $\mu_2 \leq \sigma$**, where** σ **is the (unbounded) measure on** $(-\infty, 0]$ **with density**

$$\frac{d\sigma}{dx} = \frac{\sqrt{a}}{\pi t \sqrt{|x|}} = -\frac{d}{dx} \left(\frac{2\sqrt{|ax|}}{\pi t} \right), \qquad x < 0.$$

Minimizer and g-functions

- **The constraint $\mu_2 \le \sigma$ is active only if $p < 0$, that is, if $t > t_{cr}$.**
- **The unique minimizer satisfies**

$$\operatorname{supp}(\mu_1) = \Delta_1 = [\max(0,p), q],$$
$$\operatorname{supp}(\mu_2) = (-\infty, 0],$$
$$\operatorname{supp}(\sigma - \mu_2) = \Delta_2 = (-\infty, \min(0,p)]$$

- **Explicit formulas for densities**

$$\frac{d\mu_1(x)}{dx} = \frac{1}{\pi} \left|\operatorname{Im} \zeta(x)\right| \chi_{\Delta_1}(x), \qquad \frac{d(\sigma - \mu_2)(x)}{dx} = \frac{1}{\pi} \left|\operatorname{Im} \zeta(x)\right| \chi_{\Delta_2}(x),$$

where (as before) $\zeta = \zeta(x)$ is a non-real solution of

$$x\zeta^3 - \frac{2x}{t(1-t)}\zeta^2 + \left(\frac{x}{t^2(1-t)^2} + \frac{1}{t(1-t)} - \frac{a}{t^2}\right)\zeta - \frac{1}{t^2(1-t)^2} = 0$$

Second transformation $X \mapsto U$

- **Define the g-functions** $g_j(z) = \int \log(z-s)d\mu_j(s)$ **and for certain constant matrices C_n and D_n,**

$$T(z) = C_n X(z)\,\mathrm{diag}\left(e^{-ng_1(z)}, e^{n(g_1-g_2)(z)}, e^{ng_2(z)}\right) D_n$$

- **Then we have jump matrices of the form**

$$\begin{pmatrix} e^{-2n\varphi_{1+}(x)} & x^{\alpha} & 0 \\ 0 & e^{-2n\varphi_{1-}(x)} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad x \in \Delta_1$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-2n\varphi_{2-}(x)} & 0 \\ 0 & |x|^{\alpha} & e^{-2n\varphi_{2+}(x)} \end{pmatrix}, \quad x \in \Delta_2$$

and all other jump matrices are $I +$ “exp. small”, including the one on $(p,0)$ if $t > t_{cr}$, which is where the upper constraint comes in.

Further steps

- Then we follow the general scheme of the Deift-Zhou steepest descent method
  - opening up lenses around Δ_1 and Δ_2
  - construction of global parametrix
  - local parametrices around p and q with Airy functions
  - local parametrix around 0 with Bessel functions
- Certain complications due to unboundedness of Δ_2
- Detailed analysis of critical time remains to be done