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Our activity in this direction was triggered by the paper by Dean and Majumdar, who came up with an approximate theory of large spectral fluctuations in the three Gaussian Random Matrix models – GOE, GUE and GSE.

Non-perturbative Theory of Large Spectral Deviations in GUE

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Large Deviations of Extreme Eigenvalues of Random Matrices

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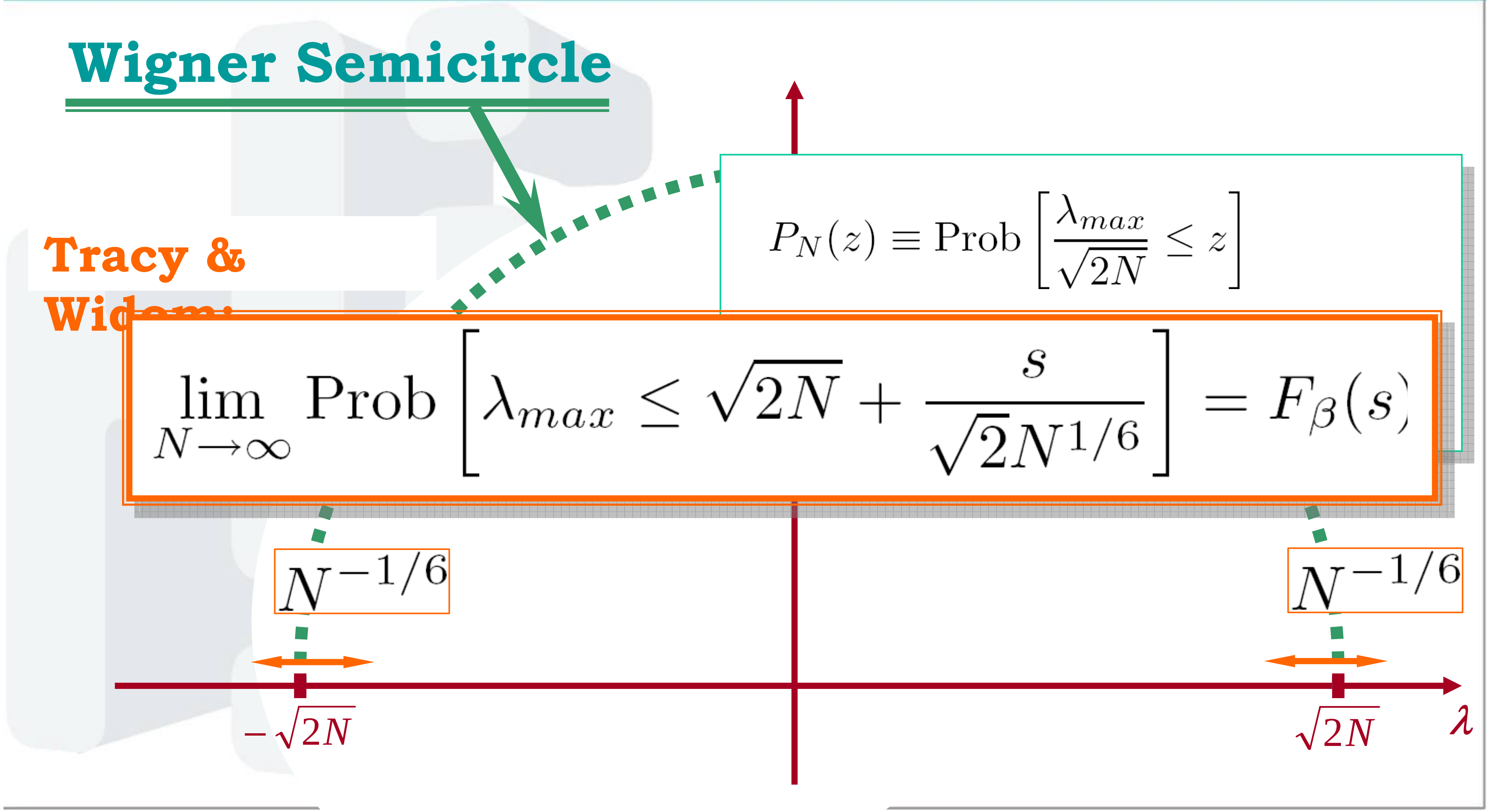
Dean & Majumdar were interested in the probability of large (and therefore rare) fluctuations of the extreme eigenvalue situated at the endpoint of the spectrum support.

This problem obviously differs from one considered by Tracy & Widom, who described the typical small fluctuations of the extreme eigenvalues that occur on the scale of order N to the power minus one sixths.

Their major result for the probability distribution function of the largest eigenvalue is given by this formula. The limit is well defined and the resulting function is the celebrated Tracy-Widom distribution, which has a representation in terms of Painleve II solution.

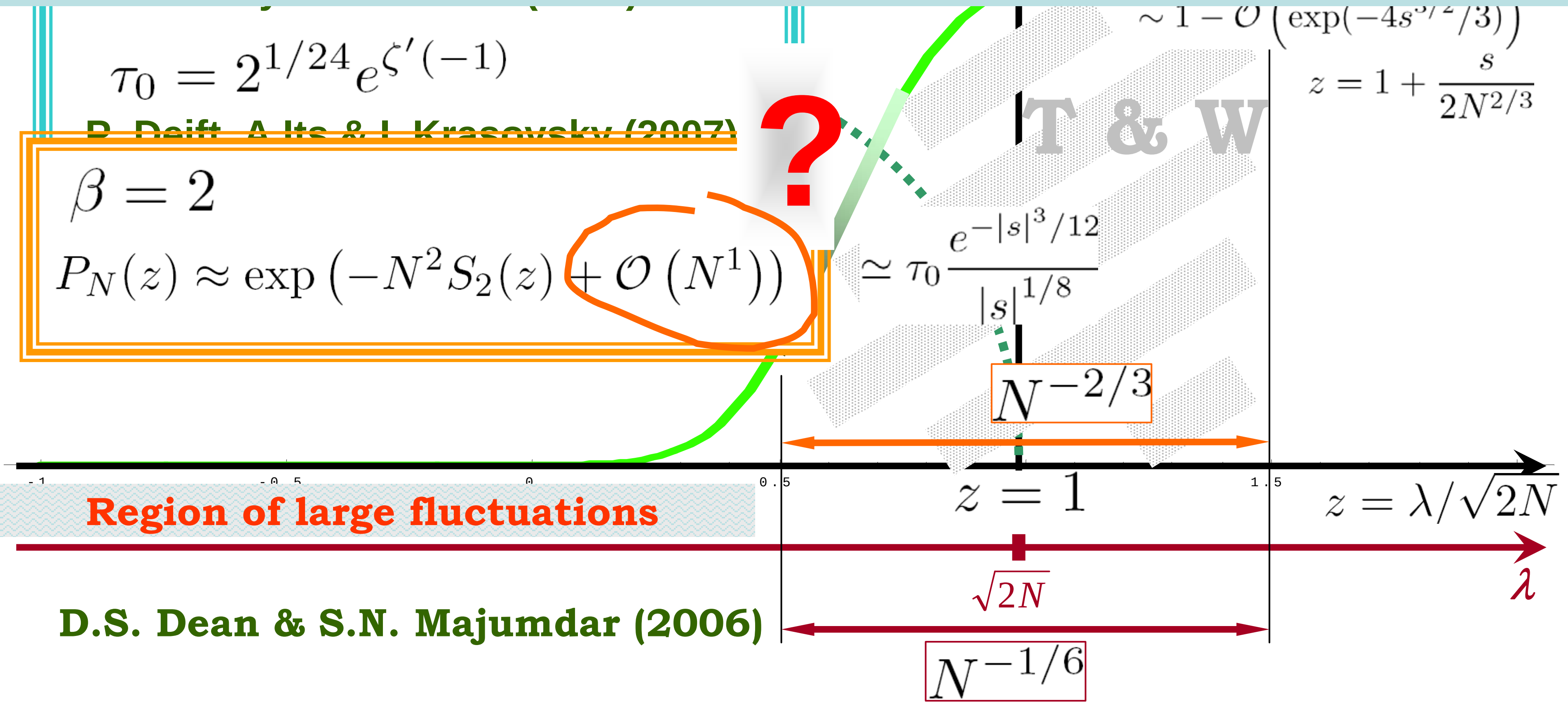


The green curve is the probability distribution of the maximal eigenvalue plotted as a function of a rescaled variable z, which differs from the conventional one by the square root of 2N, so that the point z=1 corresponds to the right end of the Wigner semicircle.

In the grey area the TW solution takes place. Its asymptotic behavior is well studied. Tau naught is a constant that was conjectured by Tracy Widom and latter re-confirmed by Deift Its and Krasovsky.

The Tracy Widom distribution describes typical fluctuations within the narrow region. On the contrary, atypical fluctuations are described by the far left tail. It is this part of the curve that was addressed in the Dean & Majumdar paper. By making use the Coulomb fluid approach they derived the distribution function with the exponential accuracy. S2 is a known smooth, positive defined function.

In addition to the term quadratic in the matrix dimension N the linear in N term was expected to exist. It is this undetermined, linear in N correction that triggered our interest to the problem.



The Major Statement of the Work

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We managed to find an asymptotically exact solution for the problem of large fluctuations and the result is in the center of this slide. The exponent here is exactly the one obtained by Dean & Majumdar but the linear in N contribution is absent. Instead, the N – dependent prefactor, $N^{-1/12}$, appears. Also, the answer contains the very same constant τ_0, that we have previously seen in the asymptotic of the Tracy & Widom solution.

D.S. Dean & S.N. Majumdar result $P_N(z) \approx \exp\left(\underline{-N^2 S_2(z)} + \mathcal{O}\left(N^1\right)\right)$

GUE$_{\mathbf{NxN}}$ $N >> 1$

$$P_N(z) = \frac{3^{1/8}\tau_0}{2^{5/24}} \frac{S_0(z)}{N^{1/12}} \exp\left(\underline{-N^2 S_2(z)}\right)$$

$$S_2(z) = \frac{1}{27}\left[(36z^2 - 2z^4) - (2z^3 + 15z)\sqrt{z^2+3} + 27\ln(\sqrt{z^2+3} - z)\right]$$

$$S_0(z) = \frac{(\sqrt{z^2+3} - z)^{1/6}}{(\sqrt{z^2+3} - 2z)^{1/8}(z^2+3)^{1/48}}$$

Probability Distribution Function via Coulomb Fluid Approach

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An absence of the linear in N term can be appreciated even within the Coulomb fluid approach. Indeed, for general beta, the desired distribution function PN which is defined by a matrix integral can approximately be rewritten as a functional integral. This representation holds in the large N-limit and each term in the action has a clear origin.

$$P_N(z) \sim \int_{-\infty}^{z} |\Delta_N(\boldsymbol{\mu})|^{\beta} \prod_{j=1}^{N} e^{-\beta N \mu_j^2} d\mu_j$$

$$N >> 1$$

$$P_N(z) \sim \int \mathcal{D}[\rho] \exp\left[-\beta N^2 \int_{-\infty}^{z} \mu^2 \rho(\mu) d\mu + \frac{\beta N^2}{2} \iint_{-\infty}^{z} d\mu d\mu' \rho(\mu)\rho(\mu') \ln|\mu - \mu'| - \left(1 - \frac{\beta}{2}\right) N \int_{-\infty}^{z} d\mu \rho(\mu) \ln[\rho(\mu)]\right]$$

$$\boxed{\rho(\mu) = \frac{1}{N}\sum_{j=1}^{N} \delta\left(\mu - \mu_j\right)}$$

Coulomb Fluid Approach

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The origin of the first term should not be explained. In the continuous limit, the Van der Monde determinant brings two contributions; the first one is due to repulsion of distant levels; the second contribution cancels the artificially induced self-interaction of levels, which is present in the double series. Finally, the integration measure gives rise to an additional term in the exponent.

F. Dyson (1962

$$P_N(z) \sim \int_{-\infty}^{z} |\Delta_N(\boldsymbol{\mu})|^\beta \prod_{j=1}^{N} e^{-\beta N \mu_j^2} d\mu_j$$

1. $e^{-\beta N \sum_j \mu_j^2} \xrightarrow{N>>1} e^{-\beta N^2 \int_{-\infty}^{z} \mu^2 \rho(\mu) d\mu}$

2. $|\Delta_N(\boldsymbol{\mu})|^\beta = e^{\beta \sum_{i<j} \ln|\mu_i - \mu_j|}$

Short-distance regularization

local level spacing $l(\mu_j) \sim 1/N\rho(\mu_j)$

$$\xrightarrow{N>>1} \approx e^{\frac{\beta}{2}\sum_{i,j} \ln|\mu_i - \mu_j| - \frac{\beta}{2}\sum_j \ln\left|\frac{1}{N\rho(\mu_j)}\right|}$$

$$e^{\frac{\beta N^2}{2} \iint_{-\infty}^{z} \ln|\mu - \mu'| \rho(\mu)\rho(\mu') d\mu d\mu' + \frac{\beta N}{2} \int_{-\infty}^{z} \rho(\mu) \ln[\rho(\mu)] d\mu}$$

3. $\int_{-\infty}^{z} \prod_{j=1}^{N} d\mu_j \xrightarrow{N>>1} \int \mathcal{D}[\rho] J(\rho)$

$$J(\rho) \sim e^{-\sum_j \ln|N\rho(\mu_j)|} \sim e^{-N\int_{-\infty}^{z} \rho(\mu) \ln[\rho(\mu)] d\mu}$$

$$\boxed{\rho(\mu) = \frac{1}{N}\sum_{j=1}^{N} \delta\left(\mu - \frac{\lambda_j}{\sqrt{2N}}\right) \qquad \int_{-\infty}^{z} \rho(\mu) d\mu = 1}$$

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Absence of the Linear in N Term. Coulomb Fluid Approach

At β=2, the linear in N term vanishes. This lead us to conclude that a linear in N term cannot be present in the Dean & Majumdar solution for β=2.

$$P_N(z) \sim \int_{-\infty}^{z} |\Delta_N(\boldsymbol{\mu})|^\beta \prod_{j=1}^{N} e^{-\beta N \mu_j^2} d\mu_j$$

$$N >> 1$$

$$P_N(z) \sim \int \mathcal{D}[\rho] \exp\Bigg[-\beta N^2 \int_{-\infty}^{z} \mu^2 \rho(\mu) d\mu + \frac{\beta N^2}{2} \iint_{-\infty}^{z} d\mu d\mu' \rho(\mu)\rho(\mu') \ln|\mu-\mu'| \cancel{- \left(1-\frac{\beta}{2}\right) N \int_{-\infty}^{z} d\mu\, \rho(\mu) \ln[\rho(\mu)]}\Bigg]$$

$$\boxed{\rho(\mu) = \frac{1}{N} \sum_{j=1}^{N} \delta\left(\mu - \mu_j\right)}$$

Non-perturbative Theory of Large Spectral Deviations

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Our consideration is based on the exact representation of the probability distribution function in terms of a solution of the fourth Painleve transcendent. This formula can be derived by using a technique of Fredholm determinant (as was done by Tracy and Widom), or with the help of theory of integrable hierarchies much in line with the works by Adler and van Moerbeke.

$$P_N(z) \equiv \mathrm{Prob}\left[\frac{\lambda_{max}}{\sqrt{2N}} \le z\right] \sim \int_{-\infty}^{z} \Delta_N^2(\boldsymbol{\mu}) \prod_{j=1}^{N} e^{-2N\mu_j^2} d\mu_j$$

$$P_N(z) = \exp\left(-\int_z^{+\infty} dx\, f_N(x)\right)$$

Painlevé IV in Chazy form

$$f_N''' + 6(f_N')^2 + 16N^2 f_N'(1-x^2) + 16N^2 x f_N = 0$$

1. **Fredholm determinant representation.**
   C.A. Tracy & H. Widom (1992)

2. **Integrable hierarchies. M. Adler & P. van Moerbeke (2001)**

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The appropriate boundary conditions were determined by Tracy and Widom who made us the result by Bassom Clarkson Hicks and McLeod.

$$P_N(z) = \exp\left(-\int_z^{+\infty} dx\, f_N(x)\right)$$

$$f_N''' + 6(f_N')^2 + 16N^2 f_N'(1-x^2) + 16N^2 x f_N = 0$$

Boundary conditions

A.P. Bassom, P.A. Clarkson, A.C. Hicks, J.B. McLeod (1992)

$$f_N(x)|_{x\to+\infty} = 0$$

$$f_N(x)|_{x\to-\infty} = N^2\left(-4x - \frac{1}{x} + \frac{1}{2x^3} - \frac{9}{16x^5} + \frac{27}{32x^7} + \mathcal{O}\left(x^{-8}\right)\right)$$

$$+ N^0\left(-\frac{1}{16x^5} + \frac{10}{32x^7} + \mathcal{O}\left(x^{-8}\right)\right)$$

1/N Expansion

Existence of the large parameter N makes it tempting to seek the solution fN in the form of the 1/N expansion. We may argue that Gamma is identically zero. The equations for Lambda and Omega together with appropriate boundary conditions follow directly from the initial equation for fN.

$$f_N''' + 6(f_N')^2 + 16N^2 f_N'(1-x^2) + 16N^2 x f_N = 0$$

$$f_N(x)|_{x\to+\infty} = 0$$

$$f_N(x)|_{x\to-\infty} = N^2\left(-4x - \frac{1}{x} + \frac{1}{2x^3} - \frac{9}{16x^5} + \frac{27}{32x^7} + \mathcal{O}\left(x^{-8}\right)\right) + N^0\left(-\frac{1}{16x^5} + \frac{10}{32x^7} + \mathcal{O}\left(x^{-8}\right)\right)$$

Ansatz

$$f_N(x) = N^2\Lambda(x) + N\Gamma(x) + \Omega(x) + \mathcal{O}\left(N^{-1}\right)$$

$$P_N(z) \simeq e^{-N^2\int_z^{+\infty}\Lambda(x)dx - N\int_z^{+\infty}\Gamma(x)dx - \int_z^{+\infty}\Omega(x)dx}$$

$$6\left(\Lambda'(x)\right)^2 + 16(1-x^2)\Lambda'(x) + 16x\Lambda(x) = 0$$

$$\Gamma(x) \equiv 0$$

$$\left(1 - x^2 + \frac{3}{4}\Lambda_1'(x)\right)\Omega'(x) + x\Omega(x) = -\frac{1}{16}\Lambda_1'''(x)$$

$\Lambda(x)$

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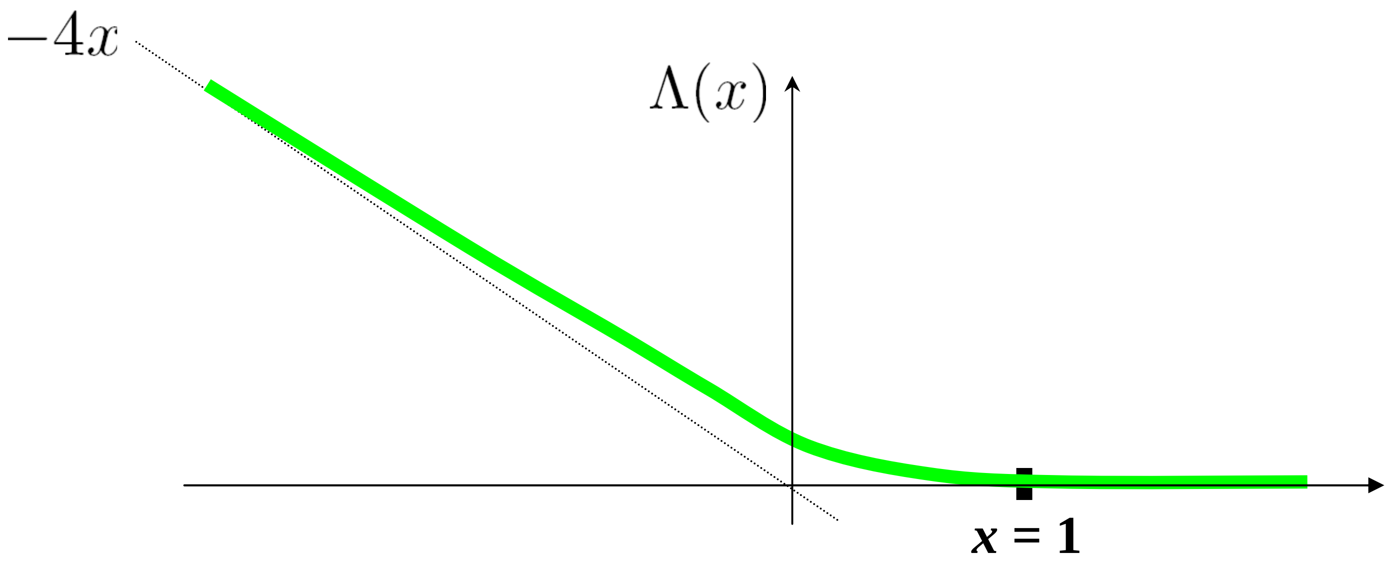
The explicit solution for Lambda, that has continuous derivative, is shown on the graph. At minus infinity it approaches the line –4x, as expected. At x large then one, Lambda vanishes.

$$P_N(z) \simeq e^{-N^2 \int_z^{+\infty} \Lambda(x)dx - \int_z^{+\infty} \Omega(x)dx}$$

$$6\left(\Lambda'(x)\right)^2 + 16(1-x^2)\Lambda'(x) + 16x\Lambda(x) = 0$$

$$\Lambda(x)|_{x\to-\infty} = -4x - \frac{1}{x} + \frac{1}{2x^3} - \frac{9}{16x^5} + \frac{27}{32x^7} + \mathcal{O}\left(x^{-8}\right) \qquad \Lambda(x)|_{x\to+\infty} = 0$$



$\Omega(x)$

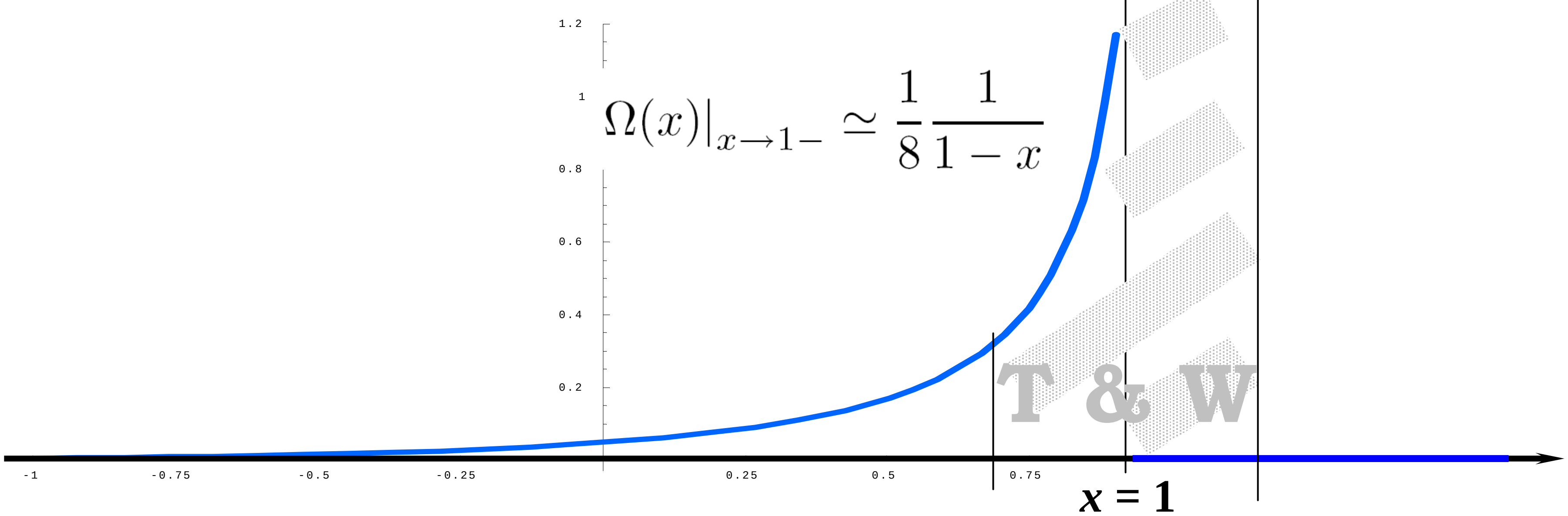
The corresponding solution for Omega is shown on this graph. This solution displays an unphysical singularity at x=1 making the probability ill defined. This is a clean signal that the 1/N ansatz breaks down in the narrow vicinity of x=1. This vicinity is nothing but the Tracy Widom domain. There the equation for fN should be solved exactly. In large N-limit this solution is the Tracy-Widom distribution.

$$P_N(z) \simeq e^{-N^2 \int_z^{+\infty} \Lambda(x)dx - \int_z^{+\infty} \Omega(x)dx}$$

$$\left(1 - x^2 + \frac{3}{4}\Lambda_1'(x)\right)\Omega'(x) + x\Omega(x) = -\frac{1}{16}\Lambda_1'''(x)$$

$$\Omega(x)|_{x\to-\infty} = -\frac{1}{16x^5} + \frac{10}{32x^7} + \mathcal{O}\left(x^{-8}\right) \qquad \Omega(x)|_{x\to+\infty} = 0$$

$$\Omega(x)|_{x\to 1-} \simeq \frac{1}{8}\frac{1}{1-x}$$



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Solution in the Tracy-Widom vicinity

$$P_N(z) = \exp\left(-\int_z^{+\infty} dx\, f_N(x)\right)$$

GUE$_{\mathbf{NxN}}$ $N >> 1$

$$f_N''' + 6(f_N')^2 + 16N^2 f_N'(1-x^2) + 16N^2 x f_N = 0$$

$$x = 1 + \frac{s}{2N^{2/3}} \qquad \varphi(s) = \frac{1}{2N^{2/3}} f(1 + s/2N^{2/3})$$

Tracy-Widom distribution

$$P_N(z)\big|_{z=1+\frac{s}{2N^{2/3}}} = \exp\left(-\int_s^{+\infty} ds'\varphi(s')\right) \equiv F_2(s)$$

$$\varphi''' + 6(\varphi')^2 - 4s\varphi' + 2\varphi = 0$$

$$\varphi(s)\big|_{s\to+\infty} = 0$$

Matching with the Tracy-Widom Distribution

$$\boxed{P_N(z) = \exp\left(-\int_z^{+\infty} dx\, f_N(x)\right)}$$

$$P_N(z)|_{z<<1} = \exp\left[-N^2\int_z^{1-cN^{-2/3}} \Lambda(x)dx - \int_z^{1-cN^{-2/3}} \Omega(x)dx - \int_{1-cN^{-2/3}}^{+\infty} f_N(x)dx\right]$$

$$c_N S_0(z)\exp\left(-N^2 S_2(z)\right)$$

$$c_N S_0(z)\exp\left(-N^2 S_2(z)\right)\Big|_{z=1-\frac{s}{2N^{2/3}}} \qquad F_2(s)$$

$$\simeq c_N \frac{2^{5/24}}{3^{1/8}} N^{1/12} \frac{e^{-|s|^3/12}}{|s|^{1/8}} \iff \simeq \tau_0 \frac{e^{-|s|^3/12}}{|s|^{1/8}}$$

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The Major Statement of the Work

GUE$_{NxN}$ $N >> 1$ $z \in$ **(Large fluctuations domain)**

$$P_N(z) = \frac{3^{1/8}\tau_0}{2^{5/24}} \frac{S_0(z)}{N^{1/12}} \exp\left(-N^2 S_2(z)\right)$$

$$S_2(z) = \frac{1}{27}\left[(\ldots + 3 - z)\right]$$

$$S_0(z) = \frac{\ldots}{(\sqrt{z}\ldots}$$

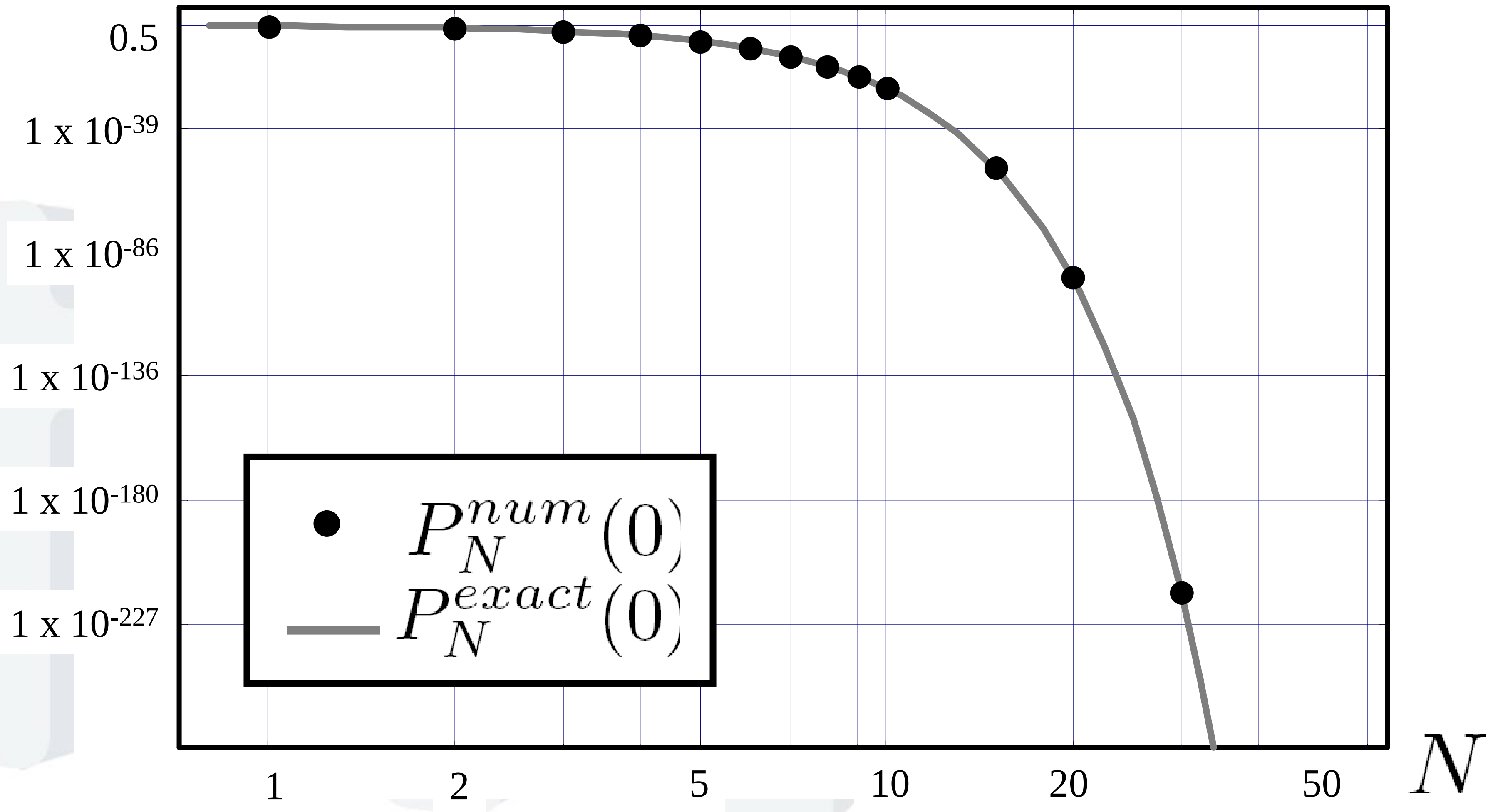
***z*=0, *N* = 10**

Numerics :	$9.95926\text{x}10^{-25}$
Dean & Majumdar:	$1.39296\text{x}10^{-24}$
Exact solution:	$9.95935\text{x}10^{-25}$

$$\delta = \left|\frac{Numerics - D\,\&\,M}{Numerics}\right| \sim 40$$

Non-perturbative Theory of Large Spectral Deviations in GUE

Numerics



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Conclusions

- **Asymptotically exact solution for the probability of Large Deviations of λ_{max} for β=2**
- No linear in N term in the exponent at β=2
- **Large fluctuations and Tracy Widom formula are two different scaling limits of the same theory**
- **Similar approach can be developed for β=1 & 4**

$$P_N(z) = \frac{3^{1/8}\tau_0}{2^{5/24}} \frac{S_0(z)}{N^{1/12}} \exp\left(-N^2 S_2(z)\right)$$