4th Brunel workshop on Random Matrix Theory
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# Anisotropic growth of random surfaces in $2+1$ dimensions


Patrik L. Ferrari
jointly with Alexei Borodin


arXiv:0804.3035 (long paper)
arXiv:0811.0682 (physics paper - short)


Bonn University

universitätbonn

http://www-wt.iam.uni-bonn.de/∼ferrari/

# Outline

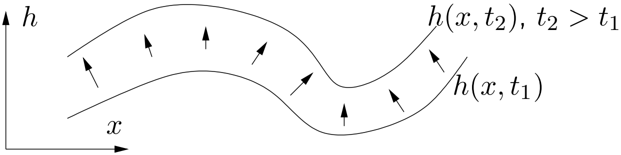
- KPZ growth in one and two dimensions
- Anisotropic KPZ
- The model
- Gaussian Free Field fluctuations

# Macroscopic evolution

- Surface described by an height function

$$h(x,t)$$

$x \in \mathbb{R}^d$ the space, $t \in \mathbb{R}$ the time




- Models with local growth

$\Rightarrow$ macroscopic growth velocity $v$ is a function of the slope only:

$$\frac{\partial h}{\partial t} = v(\nabla h)$$

The KPZ equation is

$$\frac{\partial h}{\partial t} = \nu \Delta h + \frac{\lambda}{2}(\nabla h)^2 + \eta$$

It is an equation for stochastic growth models with

- smoothing mechanism: $\Delta h$
- non-linear: $\lambda = v''(0) \neq 0$
- local noise: $\eta$

(and for $\nabla h$ small: Taylor's series)

- Smoothing mechanism ⇒ deterministic limit shape:

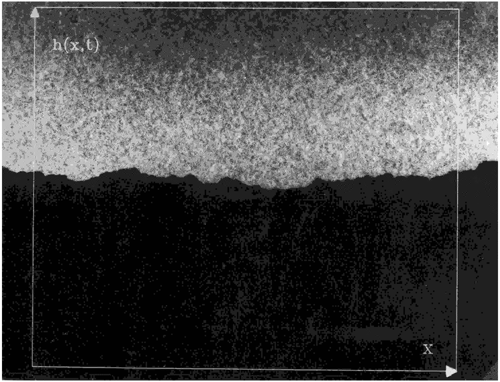
$$h_{\rm ma}(\xi) := \lim_{t\to\infty} \frac{h(\xi t, t)}{t}$$

is non-random

⇒ Focus on a mesoscopic scale, where the fluctuations are visible

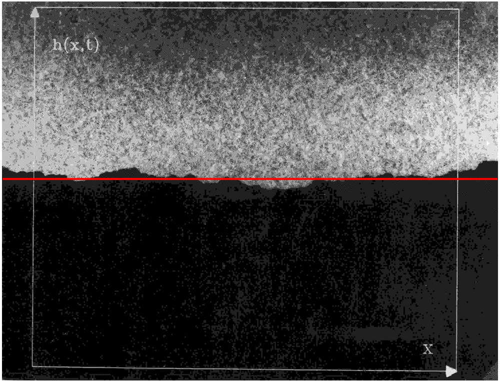
# An example in one dimension: $d = 1$

Paper burned (black) from a flame propagating from below




Q: Can we analyze the interface roughness?

Paper burned (black) from a flame propagating from below




Q: Can we analyze the interface roughness?

- Fluctuation exponent $1/3$
- Spatial correlation exponent $2/3$

$\Rightarrow$ Large time rescaling around position $\xi t$:

$$h_t^{\mathrm{resc}}(u) := \frac{h(\xi t + u t^{2/3}) - t h_{\mathrm{ma}}\left((\xi t + u t^{2/3})/t\right)}{t^{1/3}}$$

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$\Rightarrow$ Large time rescaling around position $\xi t$:

$$h_t^{\rm resc}(u) := \frac{h(\xi t + u t^{2/3}) - t h_{\rm ma}\left((\xi t + u t^{2/3})/t\right)}{t^{1/3}}$$

- Exact results in some models in the KPZ class indicate

(a) If $h''_{\rm ma}(\xi) = 0$, then

$$\lim_{t\to\infty} h_t^{\rm resc}(u) = \kappa_v \mathcal{A}_1(u/\kappa_h)$$

(b) If $h''_{\rm ma}(\xi) \neq 0$, then

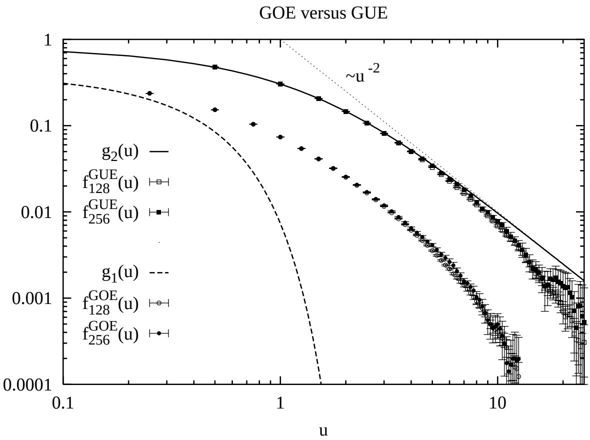
$$\lim_{t\to\infty} h_t^{\rm resc}(u) = \kappa_v \mathcal{A}_2(u/\kappa_h)$$

with $\mathcal{A}_1$, $\mathcal{A}_2$ the Airy$_1$, Airy$_2$ processes; $\kappa_v, \kappa_h$ constants.

- One-point distribution of the Airy processes:

$$\mathbb{P}(\mathcal{A}_1 \leq s) = F_{\mathrm{GOE}}(2s), \quad \mathbb{P}(\mathcal{A}_2 \leq s) = F_{\mathrm{GUE}}(s)$$

- The Airy$_2$ process describes also:
  - the (stationary) evolution of the largest eigenvalue in $\beta = 2$ Dyson's Brownian Motion
  - the border of the north polar region in the Aztec diamond
  - ...
- The Airy$_1$ process is not the limit of $\beta = 1$ Dyson's Brownian Motion! [arXiv:0806.3410]




Covariances of the Airy processes (lines) and of the GUE/GOE Dyson's Brownian Motion [arXiv:0806.3410]

- The KPZ equation in general is

$$\frac{\partial h}{\partial t} = \nu \Delta h + Q(\nabla h) + \eta$$

for some quadratic form $Q$

- In $d = 2$: two cases behaving quite differently:

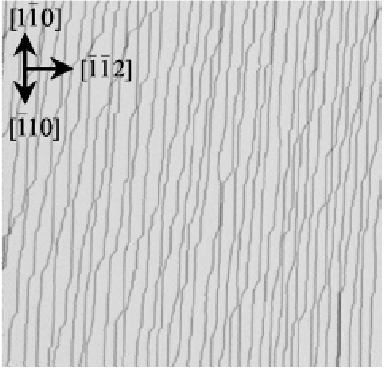
(a) Isotropic KPZ: $\mathrm{sign}(Q) = (+1, +1)$ or $\mathrm{sign}(Q) = (-1, -1)$

- Fluctuations grow as $t^{\chi}$ for some $\chi \simeq 0.240...$ (numerics)
- No analytic results

(b) Anisotropic KPZ: $\mathrm{sign}(Q) = (+1, -1)$

- The model of this talk is in this class

- Wolf [PRL '91] considered a model of vicinal surface
- small tilt with respect to a high-symmetry plane
- a model with intrinsic asymmetry




STM picture on the Si(111) with miscut of $1^\circ$ and $3^\circ$
[Yoshida, Sekiguchi, Itoha; Appl. Phys. Lett. 87, 031903 (2005)]

- Wolf realized (by one-loop expansion in renormalization) that the non-linearlity becomes irrelevant for the fluctuations:

$$\mathrm{Var}(h(\xi t, t)) \sim \ln(t), \quad t \to \infty$$

- "Irrelevant" because for the Edwards-Wilkinson model ($\lambda = 0$)

$$\frac{\partial h}{\partial t} = \nu \Delta h + \eta$$

one also has

$$\mathrm{Var}(h(\xi t, t)) \sim \ln(t), \quad t \to \infty$$

- Prähofer, Spohn '97: consider a model in the AKPZ class

(a) confirmed the Wolf's prediction

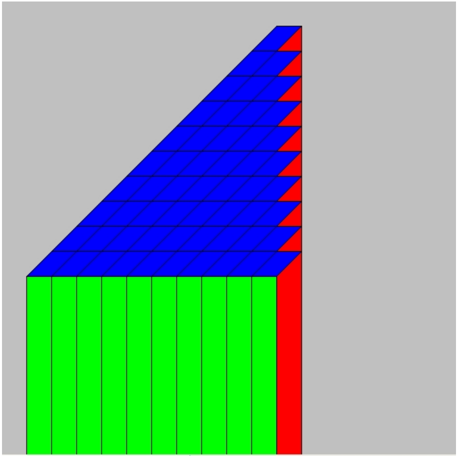
$$\mathrm{Var}(h(\xi t, t)) \sim \ln(t), \quad \text{as } t \to \infty$$

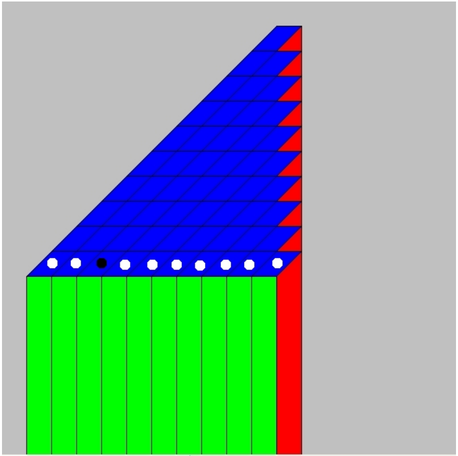
(b) determined the local height fluctuations

$$\lim_{t\to\infty} \mathrm{Var}(h(x,t) - h(x',t)) \sim \ln(|x - x'|),$$

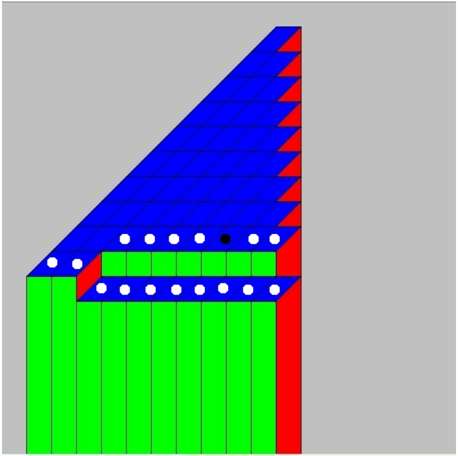
for large $|x - x'|$ (but not growing with $t$)

# The model we studied





Marked blu lozenges have independent rate one clocks



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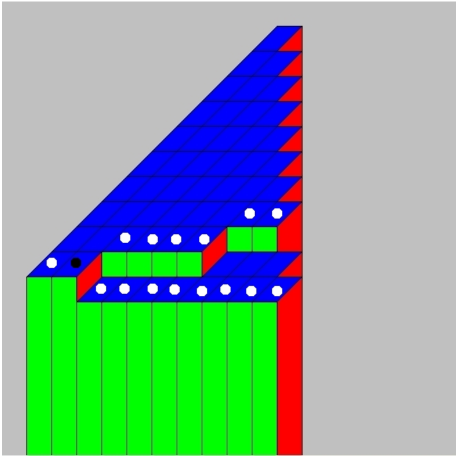

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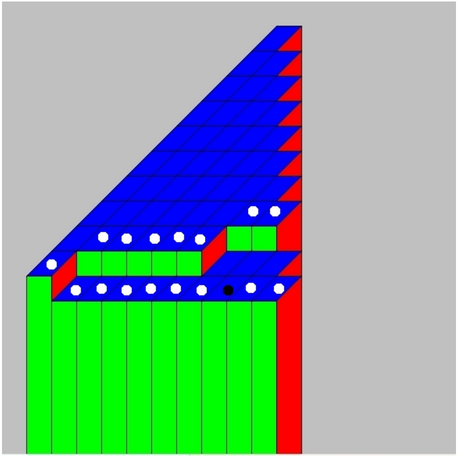

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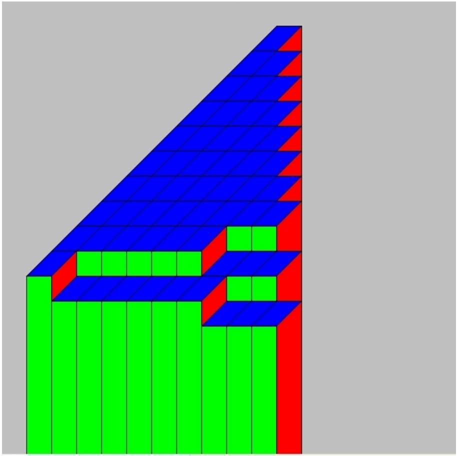
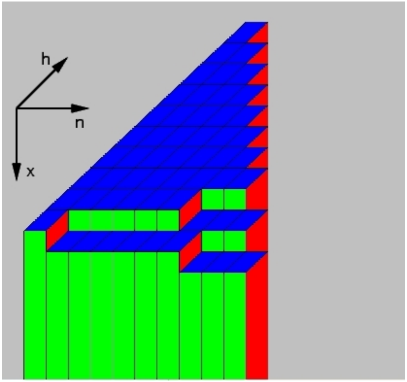





- Consider $x = x(n,h)$ describing the surface

(a) Higher density of blue lozenges at fixed $n \Rightarrow$ slower growth

(b) Higher density of blue lozenges at fixed $h \Rightarrow$ faster growth

- Height function $h = h(x, n)$
- $u_x$ = Slope along the $x$-direction
- $u_n$ = Slope along the $n$-direction

$\Rightarrow$ Speed of growth $v$ is given by

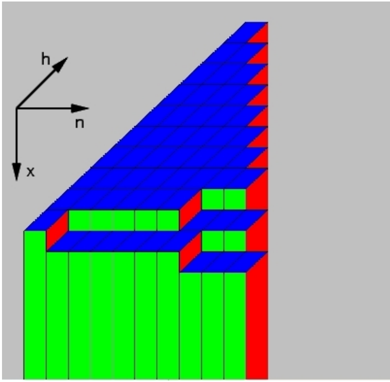
$$v(u_x, u_n) = -\frac{\sin(\pi u_x)\sin(\pi u_n)}{\pi \sin(\pi(u_x + u_n))}$$

and

$$\det(\text{Hessian}(v)) = -4\pi^2 \frac{\sin(\pi u_x)^2 \sin(\pi u_n)^2}{\sin(\pi(u_x + u_n))^4} < 0$$

- Interlacing structure of blue lozenges
  - At level $n$ there are exactly $n$ blue lozenges
  - The positions of the blue lozenges interlace

$\Rightarrow$ Same structure as the GUE minor process [math.PR/0606760]




- At fixed time $t$ and position $n$, the measure on the blue lozenges has the same form of GUE random matrices

$\Rightarrow$ We'll have determinantal correlations.

Correlation structure of the blue lozenges

### Theorem

*Consider any $N$ triples $(x_j, n_j, t_j)$ such that*

$$t_1 \leq t_2 \leq \ldots \leq t_N, \quad n_1 \geq n_2 \geq \ldots \geq n_N.$$

*Then,*

$$\mathbb{P}\big(\textit{at each } (x_j, n_j, t_j), j = 1, \ldots, N, \\ \textit{there exists a blue lozenge}\big) \\ = \det[K(x_i, n_i, t_i; x_j, n_j, t_j)]_{1 \leq i,j \leq N}$$

*for an explicit kernel $K$.*

Correlation structure of the three types of lozenges

**Theorem**

*Consider any $N$ triples $(x_j, n_j, t_j)$ such that*

$$t_1 \leq t_2 \leq \ldots \leq t_N, \quad n_1 \geq n_2 \geq \ldots \geq n_N.$$

*Then,*

$$\mathbb{P}\Big(\textit{at each } (x_j, n_j, t_j), j = 1, \ldots, N, \\ \textit{there exists a lozenge of color } c_j\Big) \\ = \det[\tilde{K}(x_i, n_i, t_i, c_i; x_j, n_j, t_j, c_j)]_{1 \leq i,j \leq N}$$

*for an explicit kernel $\tilde{K}$.*

Macroscopic parametrization:

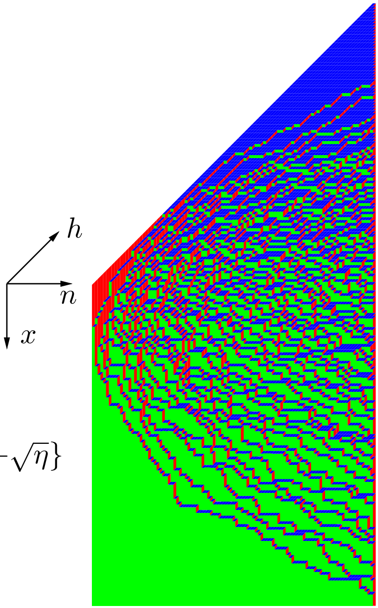
- $x = [-\eta L + \nu L]$
- $n = [\eta L]$
- $t = \tau L$

for a $L \gg 1$.

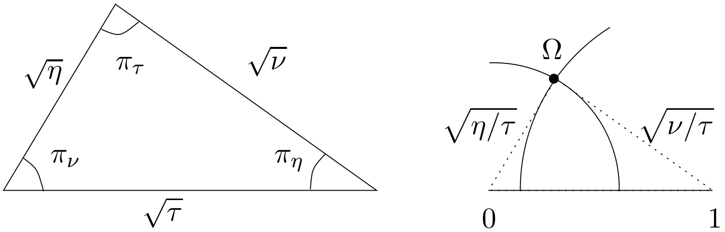
Asymptotic domain with "irregular" tiling (bordered by facets)

$$\mathcal{D} = \{(\nu, \eta, \tau), |\sqrt{\tau} - \sqrt{\eta}| < \sqrt{\nu} < \sqrt{\tau} + \sqrt{\eta}\}$$

[border of the facets: $\mathrm{Airy}_2$ process]



Bulk: $\mathcal{D} = \{(\nu, \eta, \tau) \in \mathbb{R}^3_+, |\sqrt{\tau} - \sqrt{\eta}| < \sqrt{\nu} < \sqrt{\tau} + \sqrt{\eta}\}$




Map $\Omega : \mathcal{D} \to \mathbb{H} = \{z \in \mathbb{C} | \mathrm{Im}(z) > 0\}$

$(\pi_\nu/\pi, \pi_\eta/\pi, \pi_\tau/\pi)$ are the **frequencies** of the three types of lozenge tilings: Blue for $\pi_\eta$, Red for $\pi_\tau$, Green for $\pi_\nu$

- Limit shape:

$$\bar{h}(\nu,\eta,\tau) := \lim_{L\to\infty} \frac{\mathbb{E}(h((\nu-\eta)L,\eta L,\tau L))}{L} = \int_{\nu}^{(\sqrt{\tau}+\sqrt{\nu})^2} \frac{\pi_\eta(\nu',\eta,\tau)}{\pi} \mathrm{d}\nu'$$

- The slopes are

$$\frac{\partial \bar{h}}{\partial \nu} = -\frac{\pi_\eta}{\pi}, \qquad \frac{\partial \bar{h}}{\partial \eta} = 1 - \frac{\pi_\nu}{\pi}$$

- Growth velocity:

$$\frac{\partial \bar{h}}{\partial \tau} = \frac{\sin(\pi_\nu)\sin(\pi_\eta)}{\pi \sin(\pi_\tau)} = \frac{\mathrm{Im}(\Omega)}{\pi}$$

**Theorem**

*For all $(\nu, \eta, \tau) \in \mathcal{D}$, denote $\kappa = (\nu - \eta, \eta, \tau)$. We have moment convergence of*

$$\lim_{L\to\infty} \frac{h(\kappa L) - \mathbb{E}(h(\kappa L))}{\sqrt{c \ln L}} = \xi \sim \mathcal{N}(0,1)$$

*with $c = 1/(2\pi^2)$ is independent of the macroscopic position in $\mathcal{D}$.*

### Theorem

*Consider any (disjoints) $N$ triples $\kappa_j = (\nu_j - \eta_j, \eta_j, \tau_j)$, with $(\nu_j, \eta_j, \tau_j) \in \mathcal{D}$,*

$$\tau_1 \leq \tau_2 \leq \ldots \leq \tau_N \quad \eta_1 \geq \eta_2 \geq \ldots \geq \eta_N.$$

*Set $H_L(\kappa) := \sqrt{\pi}\,(h(\kappa L) - \mathbb{E}(h(\kappa L)))$. Then,*

$$\lim_{L\to\infty} \mathbb{E}(H_L(\kappa_1)\cdots H_L(\kappa_N)) = \begin{cases} 0, & \textit{odd } N, \\ \sum_{\textit{pairings } \sigma} \prod_{j=1}^{N/2} G(\Omega_{\sigma(2j-1)}, \Omega_{\sigma(2j)}), & \textit{even } N, \end{cases}$$

*with $G(z,w) = -(2\pi)^{-1} \ln |(z-w)/(z-\bar{w})|$ is the Green function of the Laplacian on $\mathbb{H}$ with Dirichlet boundary conditions.*

# A conjecture

- Conjecture: The same result holds without the requirement

$$\tau_1 \leq \tau_2 \leq \ldots \leq \tau_N, \quad \eta_1 \geq \eta_2 \geq \ldots \geq \eta_N$$

provided all the $\Omega(\kappa_j)$'s are distincts.

- Reason: along the space-time directions with constant $\Omega$ (i.e., constant slope), the decorrelations are much slower than along any other ones

- Slow decorrelations phenomena: true for the $d = 1$ KPZ [Ferrari '08, arXiv:0806.1350]

- We considered also discrete time models, but the asymptotic analysis only for the continuous time one.
- In a particular discrete time model turns out to have the shuffling algorithm of the Aztec diamond [see Nordestam: arXiv:0802.2592]
- Allowing also deposition and not only evaporation one gets a measure at a fixed time [see Borodin-Kuan arXiv:0712.1848] $\gamma^{\pm}$ in the Plancherel measure are the jump rates for evaporation / deposition.
- Dynamics similar to an algorithm on boxed plane partitions [see Borodin-Gorin arXiv:0804.3071]