Chaotic scattering with direct processes: A generalization of Poisson's kernel for non-unitary matrices

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(V.A.G, M. Martínez-Mares, R. Méndez-Sánchez, J. Phys. A: Math. Theor. 41, 015103 (2008))

Introduction/motivation

• Different transport quantities can be written in term of the scattering matrix S associated to the system.

• RMT has been successful in describing the statistical scattering of waves through open chaotic cavities.

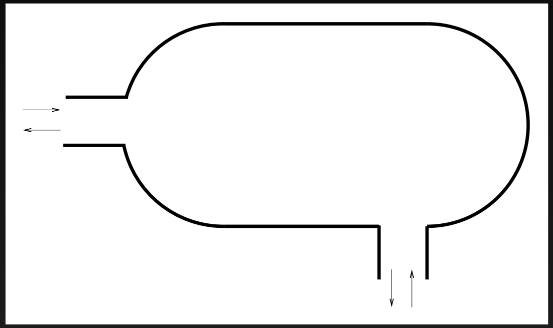
• Thus, it has been particularly useful in the study of transport quantities in billiards (chaotic quantum-dots): transmission (conductance), reflection, shot noise, admittance, etc.

• Frequently, one considers that S belongs to an ensemble of unitary scattering matrices (COE, CUE or CSE), where S is uniformly distributed.

A simple example

The distribution of the transmission for one open channel in a cavity like this one:

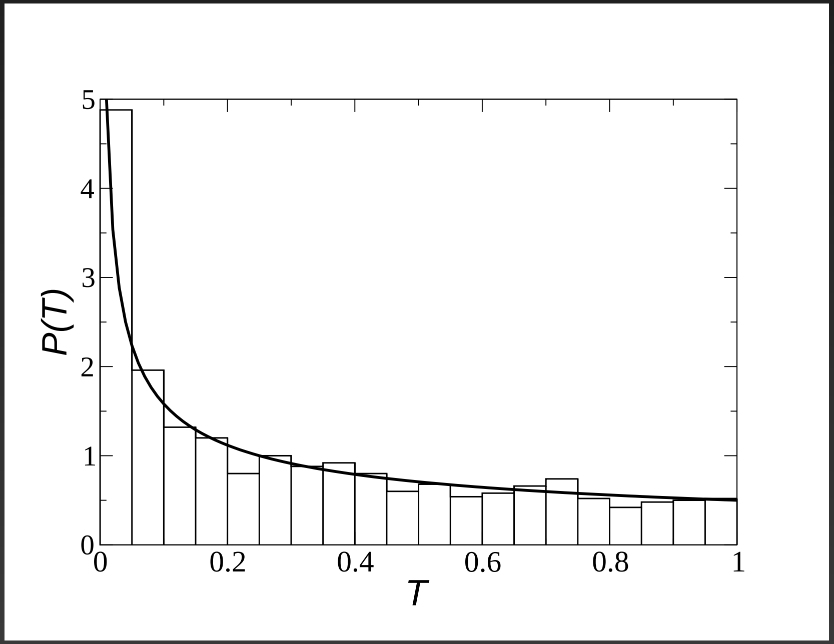
$$S = \begin{pmatrix} r & t' \\ t & r' \end{pmatrix}$$



is given by ($\beta = 1$):

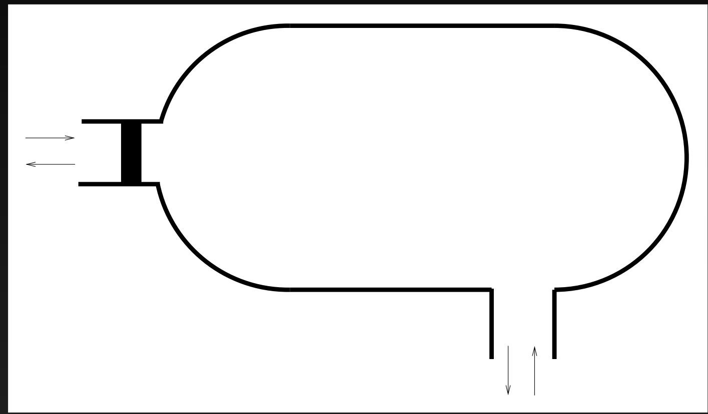
$$P(T) = \frac{1}{2\sqrt{T}},$$

where $T = \mathrm{Tr}(tt^\dagger)$



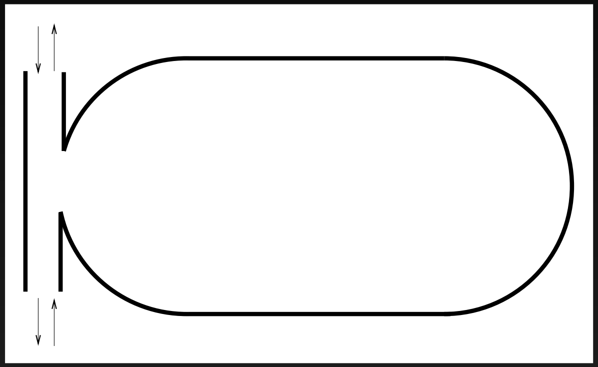
But things may be more complicated

Suppose we promote “direct reflection”



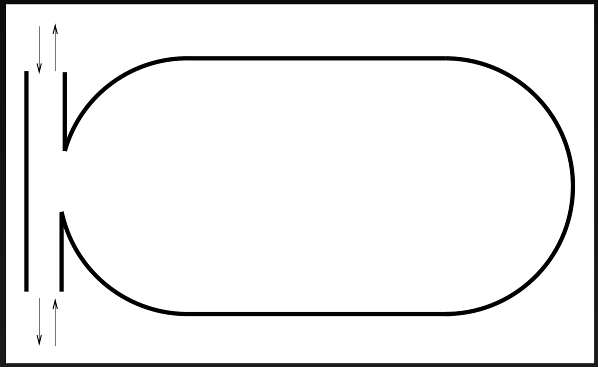
But things may be more complicated

or “direct transmission”



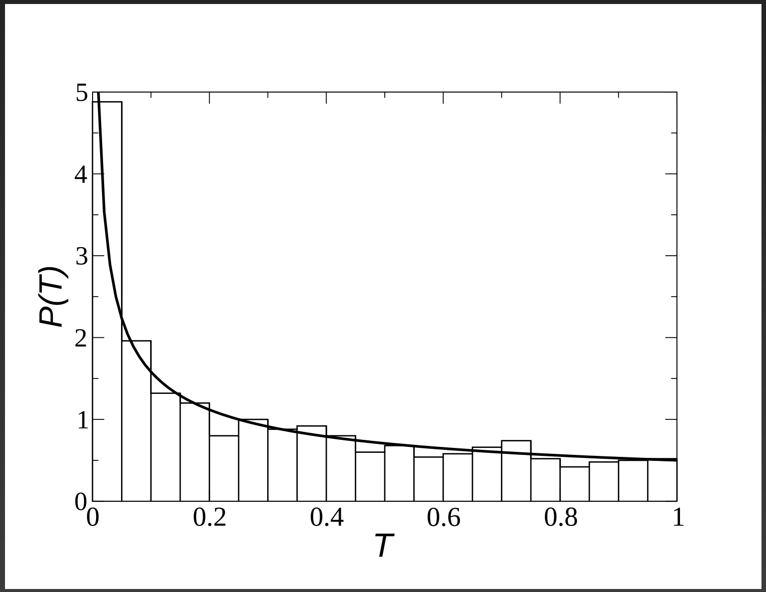
But things may be more complicated

or “direct transmission”



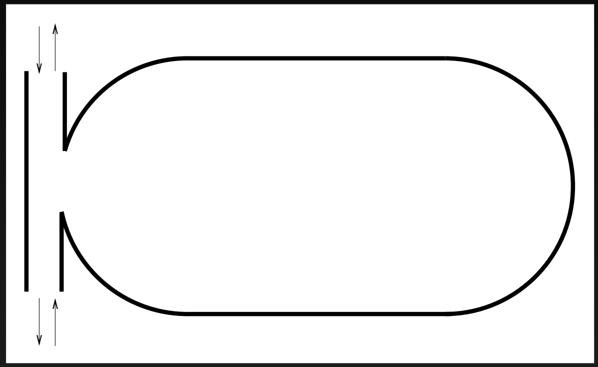
Then, the distribution may change drastically

from

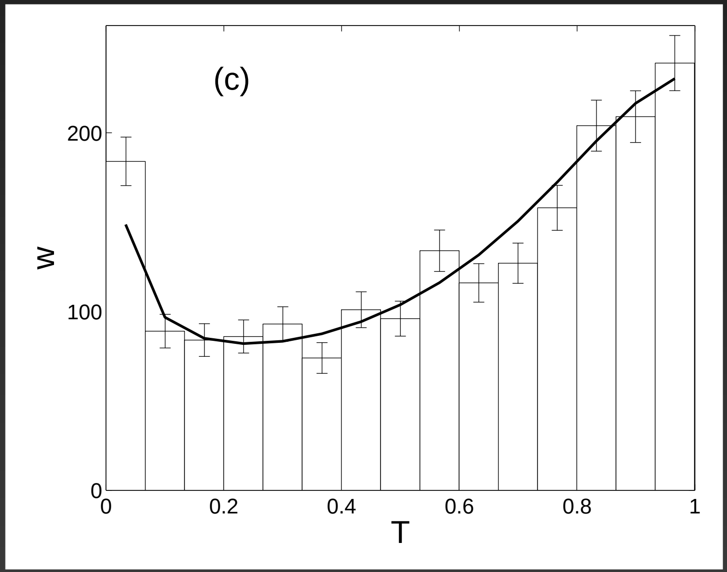


But things may be more complicated

or “direct transmission”



to



Direct processes

Chaotic scattering with direct processes is characterized by the average $\langle S\rangle$ within a maximum-entropy model. (Hua, Mello, Friedman)

Thus, in the "standard" RMT ($\langle S\rangle = 0$), the differential probability distribution of S is given by

$$dP^{(\beta)}(S) = d\mu^{(\beta)}(S),$$

while for $\langle S\rangle \neq 0$

$$dP^{(\beta)}_{\langle S\rangle}(S) = p^{(\beta)}_{\langle S\rangle}(S) d\mu^{(\beta)}(S),$$

where $d\mu$ (invariant measure) gives an equal weight to all S matrices of an ensemble and

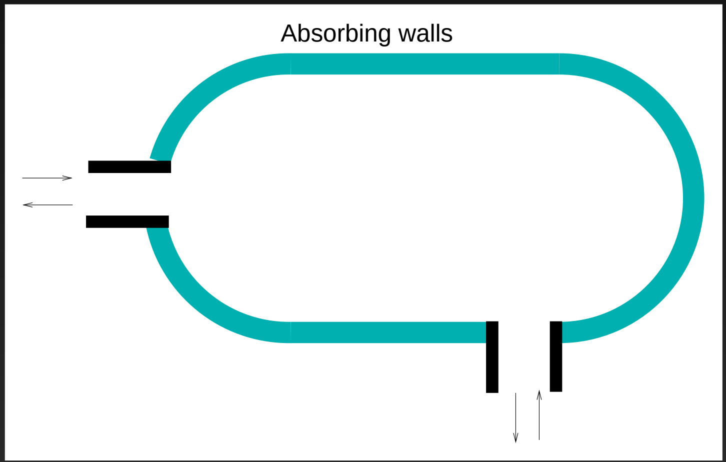
$$p^{(\beta)}_{\langle S\rangle}(S) = \frac{\left[\det\left(\mathbf{1} - \langle S\rangle\langle S\rangle^{\dagger}\right)\right]^{(\beta n+2-\beta)/2}}{\left|\det\left(\mathbf{1} - S\langle S\rangle^{\dagger}\right)\right|^{\beta n+2-\beta}} \qquad \text{Poisson's kernel}$$

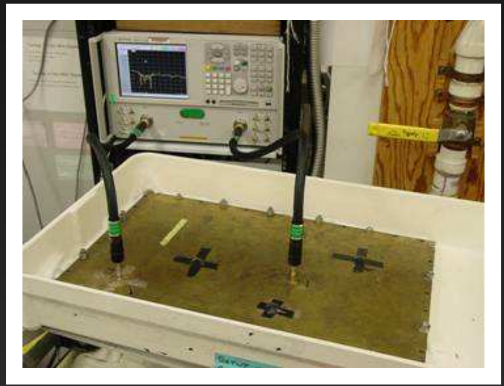
For $\langle S\rangle = 0$, $\quad p_{\langle S\rangle}(S) =$ constant

Things may be even more complicated

So far we have considered unitary S-matrices (flux conservation).

But losses are unavoidable in a number of cases: absorption in microwave experiments.





Thus, flux conservation is not satisfied ($SS^\dagger \neq 1$).

The scattering problem with losses has been studied extensively, mostly, however, in absence of direct processes.

Fyodorov, Savin, and Sommers, JPA: Math Gen., 38, 2005 and Refs. therein

Where are we ?

Unitary S-matrices and No direct processes $dP^{(\beta)}(S) = d\mu^{(\beta)}(S)$	Unitary S-matrices and Direct processes $dP^{(\beta)}_{\langle S\rangle}(S) = p^{(\beta)}_{\langle S\rangle}(S)d\mu^{(\beta)}(S)$
Non-unitary $\tilde{S}$-matrices and No direct processes $dP^{(\beta)}(\tilde{S}) = f(\tilde{S})d\mu^{(\beta)}(\tilde{S})$	Non unitary $\tilde{S}$-matrices and Direct processes $dP^{(\beta)}_{\langle \tilde{S}\rangle}(\tilde{S}) = F(\tilde{S})d\mu^{(\beta)}(\tilde{S})$

$$F(\tilde{S}) = ?$$

▷ Strategy: to reduce the problem of scattering for non unitary matrices in presence of direct processes to a problem without such prompt processes.

In this way, we can use the known results for scattering with losses in absence direct processes.

▷ How to: there is a transformation which relates non-unitary scattering matrices $\tilde{S}_0$ with $\langle \tilde{S}_0 \rangle = 0$ (absence of direct processes) to non-unitary scattering matrices $\tilde{S}$ with $\langle \tilde{S} \rangle \neq 0$ (presence of direct processes)

▷ Result: the invariant measure for systems with and without direct processes are related by

$$\boxed{d\mu^{(\beta)}(\tilde{S}_0) = \tilde{J}^{(\beta)} d\mu^{(\beta)}(\tilde{S})}$$

where

$$\boxed{\tilde{J}^{(\beta)} = \left[\frac{\left[\det\left(\mathbf{1} - \langle\tilde{S}\rangle\langle\tilde{S}\rangle^{\dagger}\right)\right]^{(\beta n+2-\beta)/2}}{|\det\left(\mathbf{1} - \tilde{S}\langle\tilde{S}\rangle^{\dagger}\right)|^{\beta n+2-\beta}} \right]^2},$$

which “is” the square of the Poisson’s kernel for unitary matrices.

The transformation is the following

$$\tilde{S}_0 = \frac{1}{\tilde{t}'_c}\left(\tilde{S} - \langle\tilde{S}\rangle\right)\frac{1}{I_n - \langle\tilde{S}\rangle^{\dagger}\tilde{S}}\tilde{t}_c^{\dagger},$$

where $\tilde{t}_c^{\dagger}\tilde{t}_c = I_n - \langle\tilde{S}\rangle^{\dagger}\langle\tilde{S}\rangle$ and $\tilde{t}'_c\tilde{t}'^{\dagger}_c = I_n - \langle\tilde{S}\rangle\langle\tilde{S}\rangle^{\dagger}$

How to obtain the previous result

Let's consider the case of symmetric non-unitary matrices $\tilde{S}_0$ and $\tilde{S}$, where these two matrices are related by the mentioned transformation.

The following is "just" algebraic manipulations...

Key steps ($\beta = 1$)

Using

$$\tilde{S} = U^T \rho U$$

U : unitary , ρ : diagonal

Differentiating

$$d\tilde{S} = U^T \delta M U$$

where

$$\delta M =$$

$$\rho(dU)U^{-1} + d\rho + (U^T)^{-1}(dU^T)\rho$$

$$d\mu(\tilde{S}) = \prod_{a \leq b} \mathrm{Re}(\delta M_{ab}) \mathrm{Im}(\delta M_{ab})$$

Similarly for $\tilde{S}_0$:

$$\tilde{S}_0 = U_0^T \rho_0 U_0,$$

$$d\tilde{S}_0 = U_0^T \delta M_0 U_0$$

and,

$$d\mu(\tilde{S}_0) = \prod_{a \leq b} \mathrm{Re}(\delta M_{0_{ab}}) \mathrm{Im}(\delta M_{0_{ab}})$$

Now, from the transformation

$$d\tilde{S}_0 = A^T d\tilde{S} A,$$

with $A = [1 - \langle S^* \rangle S]^{-1} \tilde{t}_c$

Therefore,

$$\begin{aligned} \delta M_0 &= \underbrace{\left[(U_0^T)^{-1} A^T U^T\right]} \delta M \underbrace{\left[U A U_0^{-1}\right]} \\ &= B^T \delta M B \end{aligned}$$

and the Jacobian, $\tilde{J}_{ab}$, relates the $\delta M_{0_{ab}}$ and δM_{ab} through

$$\mathrm{Re}(\delta M_{0_{ab}})\mathrm{Im}(\delta M_{0_{ab}}) = \tilde{J}_{ab}\mathrm{Re}(\delta M_{ab})\mathrm{Im}(\delta M_{ab}).$$

It turns out that the Jacobian between the invariant measures, $\tilde{J} = \prod_{a \leq b} \tilde{J}_{ab}$, is given by

$$\begin{aligned} \tilde{J} &= |\det(B^T)^{(n+1)/2} \det(B^{(n+1)/2})|^2 \\ &= |\det(A^2)^{(n+1)/2}|^2 \\ &= \left[\frac{\det(\tilde{t}_c \tilde{t}_c)^{\frac{n+1}{2}}}{|\det(1 - \tilde{S}\langle\tilde{S}\rangle^\dagger)|^{n+1}}\right]^2 = \left[\frac{\det(\mathbf{1} - \langle\tilde{S}\rangle\langle\tilde{S}\rangle^\dagger)^{\frac{n+1}{2}}}{|\det(\mathbf{1} - \tilde{S}\langle\tilde{S}\rangle^\dagger)|^{n+1}}\right]^2 \end{aligned}$$

In conclusion

$$d\mu^{(\beta)}(\tilde{S}_0) = \tilde{J}^{(\beta)} d\mu^{(\beta)}(\tilde{S})$$

$$\tilde{J}^{(\beta)} = \left[\frac{\left[\det\left(\mathbf{1} - \langle\tilde{S}\rangle\langle\tilde{S}\rangle^\dagger\right)\right]^{(\beta n+2-\beta)/2}}{\left|\det\left(\mathbf{1} - \tilde{S}\langle\tilde{S}\rangle^\dagger\right)\right|^{\beta n+2-\beta}} \right]^2,$$

In words,
if a non-unitary scattering matrix $\tilde{S}_0$ is uniformly distributed in the space of non-unitary scattering matrices, a non-unitary scattering matrix $\tilde{S}$ obtained from $\tilde{S}_0$ through the transformation is distributed according to $\tilde{J}^{(\beta)}$.

A simple example: one channel

$$\tilde{S} = \sqrt{R}\,\mathrm{e}^{\mathrm{i}\theta} \quad \text{with} \quad \mathrm{d}\mu_\beta(\tilde{\mathrm{S}}) = \mathrm{dR}\,\frac{\mathrm{d}\theta}{2\pi}.$$

A non uniform distribution of $\tilde{S}$ is constructed from as

$$\mathrm{dP}(\tilde{\mathrm{S}}) = \mathrm{p}(\mathrm{R},\theta)\mathrm{dR}\,\frac{\mathrm{d}\theta}{2\pi}.$$

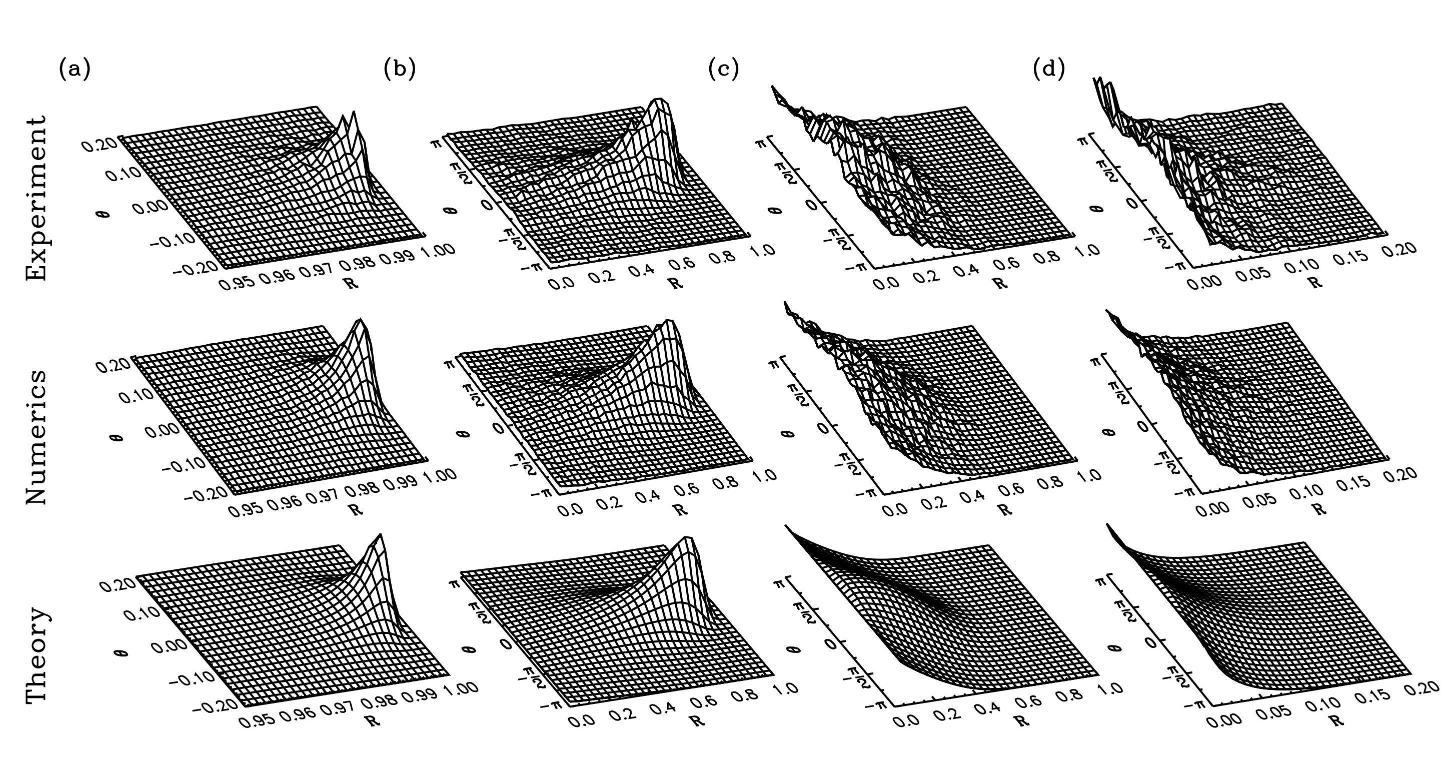
If $\tilde{S}_0 = \sqrt{R_0}\,\mathrm{e}^{\mathrm{i}\theta_0}$ is associated to chaotic cavities with losses in the absence of direct processes, $p_0(R_0,\theta_0) = p_0(R_0)$, where $p_0(R_0)$ is known . In the presence of direct processes, according to $\tilde{J}$:

$$\mathrm{dR}_0\,\frac{\mathrm{d}\theta_0}{2\pi} = \left(\frac{1-|\langle\tilde{\mathrm{S}}\rangle|^2}{|1-\tilde{\mathrm{S}}\langle\tilde{\mathrm{S}}\rangle^*|^2}\right)^2 \mathrm{dR}\,\frac{\mathrm{d}\theta}{2\pi}.$$

Multiplying by $p_0(R_0(R,\theta))$ and comparing the RHS with $\mathrm{dP}(\tilde{\mathrm{S}})$, we find

$$p(R,\theta) = \left(\frac{1-|\langle\tilde{S}\rangle|^2}{|1-\tilde{S}\langle\tilde{S}\rangle^*|^2}\right)^2 p_0(R_0(R,\theta)).$$

Kuhl, et. al.PRL 94, 2005
Savin, et. al., JETP L 62 2005



Kuhl, et. al.PRL 94, 2005

Summary

∘ We have reduced the problem of scattering in the presence of direct processes to case of absence of such processes for $n \times n$ *non-unitary* scattering matrices. In our theoretical framework, the direct processes are characterized by the average $\langle \tilde{S} \rangle$.

∘ We have used a transformation to map an ensemble of matrices $\tilde{S}$ with $\langle \tilde{S} \rangle \neq 0$ to an ensemble of $\tilde{S}_0$ scattering matrices with $\langle \tilde{S}_0 \rangle = 0$, i.e., $\tilde{S}$ and $\tilde{S}_0$ describe a system in the presence and in the absence of direct processes, respectively.

∘ The Jacobian $\tilde{J}_\beta$ of the transformation turns out to be the square of the known Poisson kernel for general complex, symmetric, and self-dual S-matrices, in analogy to the three symmetries in Dyson's scheme $\beta = 2, 1,$ and 4.

∘ Thus, if $\tilde{S}_0$ is uniformly distributed in the space of non-unitary scattering matrices, $\tilde{S}$ obtained from $\tilde{S}_0$ through transformation is distributed according to $\tilde{J}_\beta$.

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It is convenient to separate the real and imaginary parts of δM_0 and δM to obtain the Jacobian of the transformation

$$\begin{aligned}\mathrm{Re}(\delta M_{0ab}) &= \sum_{c,d=1}^{n} \mathrm{Re}(B'_{ac}B_{db})\mathrm{Re}(\delta M_{cd}) + \sum_{c,d=1}^{n} \mathrm{Re}(\mathrm{i}\mathrm{B}'_{\mathrm{ac}}\mathrm{B}_{\mathrm{db}})\mathrm{Im}(\delta \mathrm{M}_{\mathrm{cd}}),\\ \mathrm{Im}(\delta M_{0ab}) &= \sum_{c,d=1}^{n} \mathrm{Im}(B'_{ac}B_{db})\mathrm{Re}(\delta M_{cd}) + \sum_{c,d=1}^{n} \mathrm{Im}(\mathrm{i}\mathrm{B}'_{\mathrm{ac}}\mathrm{B}_{\mathrm{db}})\mathrm{Im}(\delta \mathrm{M}_{\mathrm{cd}}).\end{aligned}$$

Next, we calculate the Jacobian $\tilde{J}_{ab}^{(\beta)}$ of the transformation which relates the real and imaginary parts of the independent elements of δM_0 with those of δM as

$$\mathrm{Re}(\delta M_{0ab})\mathrm{Im}(\delta M_{0ab}) = \tilde{J}_{ab}^{(\beta)}\,\mathrm{Re}(\delta M_{ab})\mathrm{Im}(\delta M_{ab}).$$

Let B and B'

$$\begin{aligned} B_{ab} &= \lambda_a \, \delta_{ab}, \\ B'_{ab} &= \lambda'_a \, \delta_{ab}, \end{aligned}$$

where λ_a's and λ'_a's are complex numbers. Therefore

$$\begin{aligned} \mathrm{Re}(\delta M_{0ab}) &= \mathrm{Re}(\lambda'_a \lambda_b) \, \mathrm{Re}(\delta M_{ab}) - \mathrm{Im}(\lambda'_a \lambda_b) \, \mathrm{Im}(\delta M_{ab}) \\ \mathrm{Im}(\delta M_{0ab}) &= \mathrm{Im}(\lambda'_a \lambda_b) \, \mathrm{Re}(\delta M_{ab}) + \mathrm{Re}(\lambda'_a \lambda_b) \, \mathrm{Im}(\delta M_{ab}). \end{aligned}$$

From these two equations, the Jacobian $\tilde{J}^{(\beta)}_{ab}$ is given by

$$\tilde{J}^{(\beta)}_{ab} = [\mathrm{Re}(\lambda'_a \lambda_b)]^2 + [\mathrm{Im}(\lambda'_a \lambda_b)]^2 = |\lambda'_a \lambda_b|^2$$