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Spectra of Sparse Random Matrices and Localization on Random Graphs

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Overview

- Look at spectra of sparse symmetric random matrices
  - Follow replica formulation of Edwards and Jones (76), Rodgers and Bray (88)
  - Use techniques recently developed for StatMech of finitely coordinated random systems
  - Use different representation of replica symmetric ansatz
  - Identify DOS of localized and extended states
  - Deconvolution: local DOS of vertices with different coordination
- Explore for various ensembles
- Some details in
  - RK, J Phys A41, 295002, (2008), cond-mat/0803.2886
  - T. Rogers, I. Perez Castillo, RK, and K. Takeda Phys Rev E 78, 031116 (2008), cond-mat/0803.1553

Spectral Density and Resolvent

- Spectral density of random matrix M from resolvent

$$\overline{\rho(\lambda)} = \lim_{N\to\infty} \frac{1}{\pi N} \text{Im Tr } \overline{[\lambda_\varepsilon I - M]^{-1}} \,, \qquad \lambda_\varepsilon = \lambda - i\varepsilon$$

- express (S F Edwards & R C Jones, JPA, 1976) as

$$\begin{aligned} \overline{\rho(\lambda)} &= \lim_{N\to\infty} \frac{1}{\pi N} \text{ Im } \frac{\partial}{\partial\lambda} \text{ Tr } \overline{\ln\left[\lambda_\varepsilon I - M\right]} \\ &= \lim_{N\to\infty} -\frac{2}{\pi N} \text{ Im } \frac{\partial}{\partial\lambda} \overline{\ln Z_N} \,, \end{aligned}$$

where Z_N is a Gaussian integral:

$$Z_N = \int \prod_{i=1}^{N} \frac{\mathrm{d}u_i}{\sqrt{2\pi/i}} \, \exp\left\{ -\frac{i}{2} \sum_{i,j} u_i(\lambda_\varepsilon \delta_{ij} - M_{ij}) u_j \right\}$$

Sparse Random Matrices

- Sparse symmetric matrix M given, e.g. by

$$M_{ij} = c_{ij} K_{ij}$$

with $\{c_{ij}\}$ adjacency matrix of a random graph. E. g.

$$c_{ij} = \begin{cases} 0 & ;\text{with prob} \quad 1 - \frac{c}{N} \\ 1 & ;\text{with prob} \quad \frac{c}{N} \end{cases}$$

$\equiv$ Posisssonian (Erdös Renyi) random graph.
Others: regular, scale-free, small-world . . .

- Distribution of K_{ij} arbitrary
(Gaussian, bimodal, non-random . . .)

- Exploit StatMech techniques for sparsely coordinated amorphous systems. (RK, J van Mourik, M Weigt, A Zippelius J Phys A,2007)

Performing the Average — Replica Method

- Replica identity

$$\overline{\ln Z_N} = \lim_{n\to 0} \frac{1}{n} \ln \overline{Z_N^n}$$

- For integer n, Z_N^n is partition function of n identical copies of the system (n-th power of Gaussian integral)

$$\begin{aligned}\overline{Z_N^n} \;=\; \int \prod_{ia} \frac{\mathrm{d}u_{ia}}{\sqrt{2\pi/i}} \exp\Bigg\{ & -\frac{i}{2}\lambda_\varepsilon \sum_{i,a} u_{ia}^2 \\ & + \frac{c}{2N}\sum_{ij}\left(\left\langle \exp\left(iK\sum_a u_{ia}u_{ja}\right)\right\rangle_K - 1\right)\Bigg\}\end{aligned}$$

- Decoupling of sites by introducing the replicated density

$$\rho(\boldsymbol{u}) = \frac{1}{N}\sum_i \prod_a \delta(u_a - u_{ia})$$

- Enforce definition via (functional) δ-distribution

$$1 = \int \mathcal{D}\rho\mathcal{D}\hat{\rho}\ \exp\left\{-i\int \mathrm{d}\boldsymbol{u}\hat{\rho}(\boldsymbol{u})\Big(N\rho(\boldsymbol{u}) - \sum_i \prod_a \delta(u_a - u_{ia})\Big)\right\}$$

- Gives

$$\overline{Z_N^n} \;=\; \int \mathcal{D}\rho \int \mathcal{D}\hat{\rho}\ \exp\left\{N\left[\frac{c}{2}\int \mathrm{d}\rho(\boldsymbol{u})\mathrm{d}\rho(\boldsymbol{v})\left(\left\langle \exp\left(iK\sum_a u_a v_a\right)\right\rangle_K - 1\right)\right.\right.$$
$$\left.\left. - \int \mathrm{d}\boldsymbol{u}\, i\hat{\rho}(\boldsymbol{u})\rho(\boldsymbol{u}) + \ln \int \prod_a \frac{\mathrm{d}u_a}{\sqrt{2\pi/i}} \exp\left(i\,\hat{\rho}(\boldsymbol{u}) - \frac{i}{2}\lambda_\varepsilon \sum_a u_a^2\right)\right]\right\}$$

- Evaluation of $N^{-1}\ln\overline{Z_N^n}$ by saddle point method

- Stationarity w.r.t. ρ and $\widehat{\rho}$

$$\frac{\delta}{\delta\rho(\boldsymbol{u})}: \qquad i\widehat{\rho}(\boldsymbol{u}) = c\int \mathrm{d}\rho(\boldsymbol{v})\left(\left\langle \exp\left(iK\sum_a u_a v_a\right)\right\rangle_K - 1\right) \qquad (*)$$

$$\frac{\delta}{\delta\widehat{\rho}(\boldsymbol{u})}: \qquad \rho(\boldsymbol{u}) = \frac{\exp\left(i\,\widehat{\rho}(\boldsymbol{u}) - \frac{i}{2}\lambda_\varepsilon \sum_a u_a^2\right)}{\int \mathrm{d}\boldsymbol{u}\exp\left(i\,\widehat{\rho}(\boldsymbol{u}) - \frac{i}{2}\lambda_\varepsilon \sum_a u_a^2\right)} \qquad (**)$$

- Problem: $n \longrightarrow 0$ limit. (GJ Rodgers, AJ Bray, PRB 37, 1988)
  Ansatz: permutation & rotational symmetry in replica space

$$i\,\widehat{\rho}(\boldsymbol{u}) = cg(|\boldsymbol{u}|)$$

  exploit to perform ‘angular integrals in (*),(**)

- For $K \in \{\pm 1\}$ get

$$g(u) = -u \int_0^\infty \mathrm{d}v \exp\left[cg(v) - \frac{i}{2}\lambda_\varepsilon v^2\right] J_1(uv) \ , \ \text{as} \quad n \to 0$$

Independent SuSy derivation (YV Fyodorov, AD Mirlin, JPA 24, 1991)

- Rodgers-Bray Equation extremely difficult to analyze.
- Here: different representation of permutation & rotational symmetry. Superpositions of Gaussians:

$$\rho(\boldsymbol{u}) \;=\; \int \mathrm{d}\pi(\omega) \prod_a \frac{\exp\left[-\frac{\omega}{2}u_a^2\right]}{Z(\omega)}$$

$$i\hat{\rho}(\boldsymbol{u}) \;=\; c\int \mathrm{d}\hat{\pi}(\hat{\omega}) \prod_a \frac{\exp\left[-\frac{\hat{\omega}}{2}u_a^2\right]}{Z(\hat{\omega})}$$

$\Leftrightarrow$ solve (*),(**) in terms of an integral transformation

- Get saddle point equations for π and $\hat{\pi}$

Population Dynamics

- Self-consistency equations for π and $\widehat{\pi}$: pair of non-linear integral equations

$$\begin{aligned}\widehat{\pi}(\widehat{\omega}) &= \int \mathrm{d}\pi(\omega)\ \Big\langle \delta\Big(\widehat{\omega} - \widehat{\Omega}(\omega, K)\Big)\Big\rangle_K \\ \pi(\omega) &= \sum_{k\geq 1} \frac{k}{c} p_c(k) \int \prod_{\ell=1}^{k-1} \mathrm{d}\widehat{\pi}(\widehat{\omega}_\ell)\ \delta\left(\omega - \Omega_{k-1}\right)\end{aligned}$$

with

$$\widehat{\Omega}(\omega, K) = \frac{K^2}{\omega}\ , \qquad \Omega_{k-1} = i\lambda_\varepsilon + \sum_{\ell=1}^{k-1} \widehat{\omega}_\ell$$

- Structure suggests solving via stochastic population based algorithm; note: get complex $\omega, \widehat{\omega}$, but Re$(\omega) \geq 0$, Re$(\widehat{\omega}) \geq 0$ selfconsistently in population.

Spectral Density

- Spectral density from solution (using $\{\hat{\boldsymbol{\omega}}\}_k = \sum_{\ell=1}^{k} \widehat{\omega}_\ell$)

$$\overline{\rho(\lambda)} = \frac{1}{\pi} \sum_{k=0}^{\infty} p_c(k) \int \prod_{\ell=1}^{k} \mathrm{d}\widehat{\pi}(\widehat{\omega}_\ell) \, \frac{\mathrm{Re}\{\hat{\boldsymbol{\omega}}\}_k + \varepsilon}{(\mathrm{Re}\{\hat{\boldsymbol{\omega}}\}_k + \varepsilon)^2 + (\lambda + \mathrm{Im}\,\{\hat{\boldsymbol{\omega}}\}_k)^2}$$

- With

$$P(a,b) := \sum_k p_c(k) \int \prod_{\ell=1}^{k} \mathrm{d}\widehat{\pi}(\widehat{\omega}_\ell) \, \delta\left(a - \mathrm{Re}\,\{\widehat{\omega}\}_k\right) \delta\left(b - \mathrm{Im}\,\{\widehat{\omega}\}_k\right) \, ,$$

get

$$\overline{\rho(\lambda)} = \int \frac{\mathrm{d}a \; \mathrm{d}b}{\pi} \, P(a,b) \, \frac{a + \varepsilon}{(a+\varepsilon)^2 + (b+\lambda)^2} \, .$$

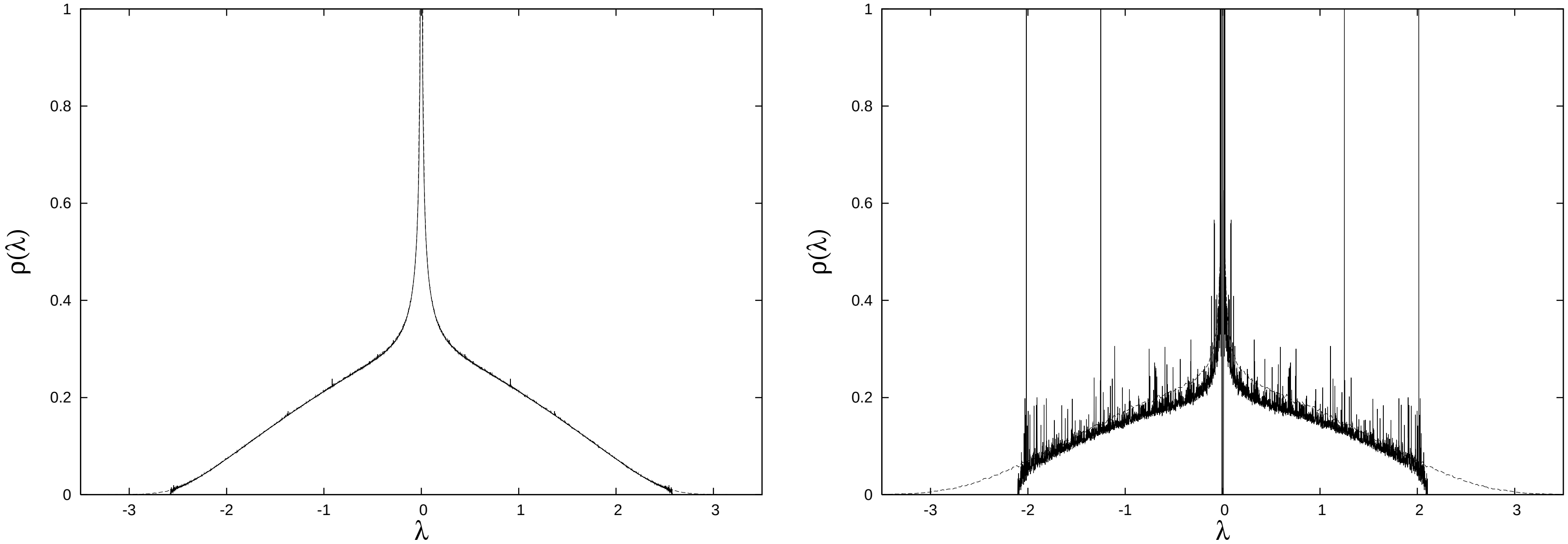
- Note: singular nature of integrand for $a = 0$, as $\varepsilon \to 0$:

$$P(a,b) = P_0(b)\delta(a) + \tilde{P}(a,b)$$

- Identify localized DOS $\qquad \overline{\rho_{\mathrm{loc}}(\lambda)} = P_0(-\lambda)$

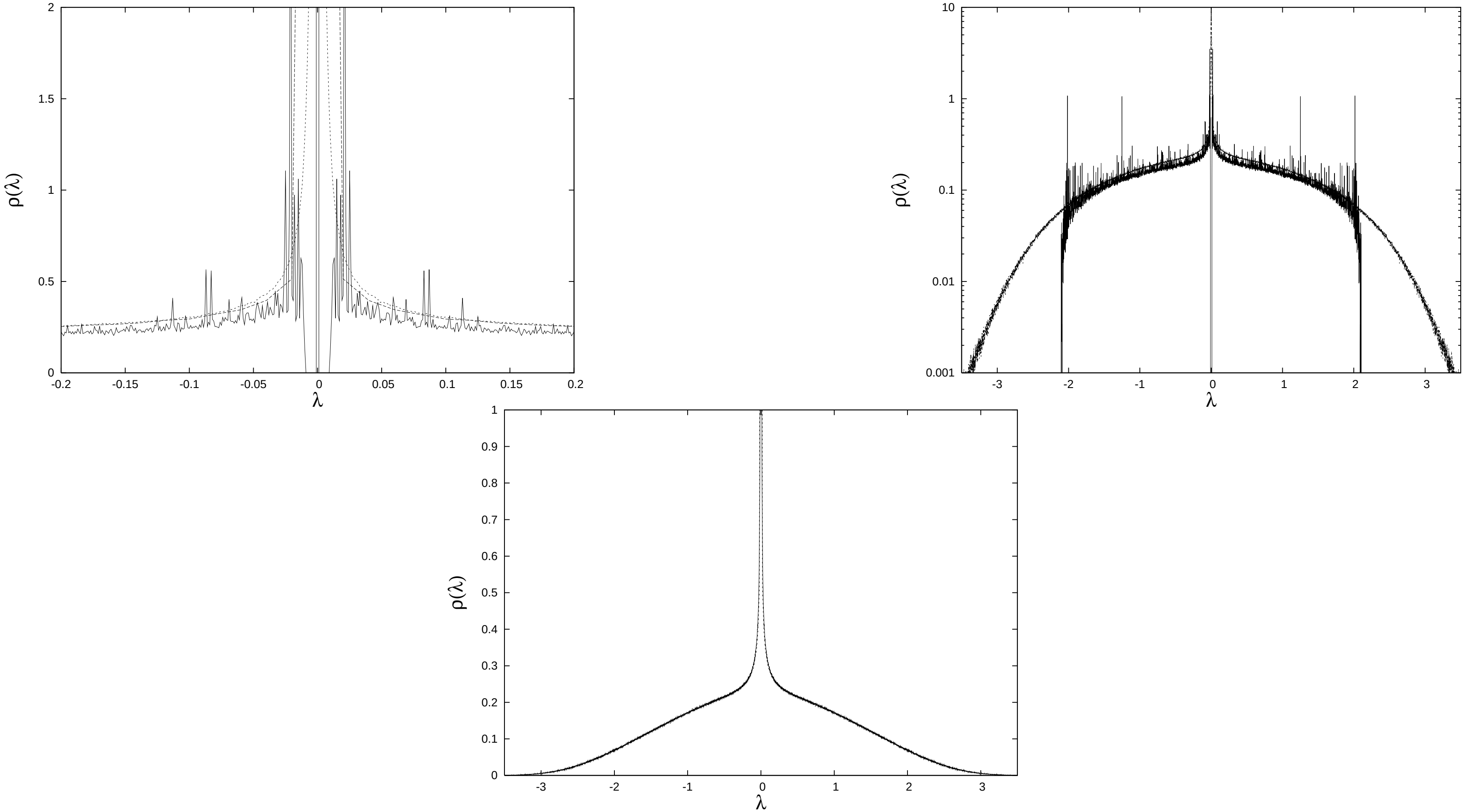
(R Abou-Chacra, PW Anderson, DJ Thouless, JPC 6, 1973)

Results — Poisson Random Graphs



Spectral densities for $\langle K_{ij}^2 \rangle = 1/c$, on Poissonian random graphs with $c = 4$ (left), and $c = 2$ (right) using $\varepsilon = 10^{-300}$ (full line); in both panels: numerical diagonalization results for graphs of size $N = 2000$ (dashed).

More on the Posisson $c = 2$ case

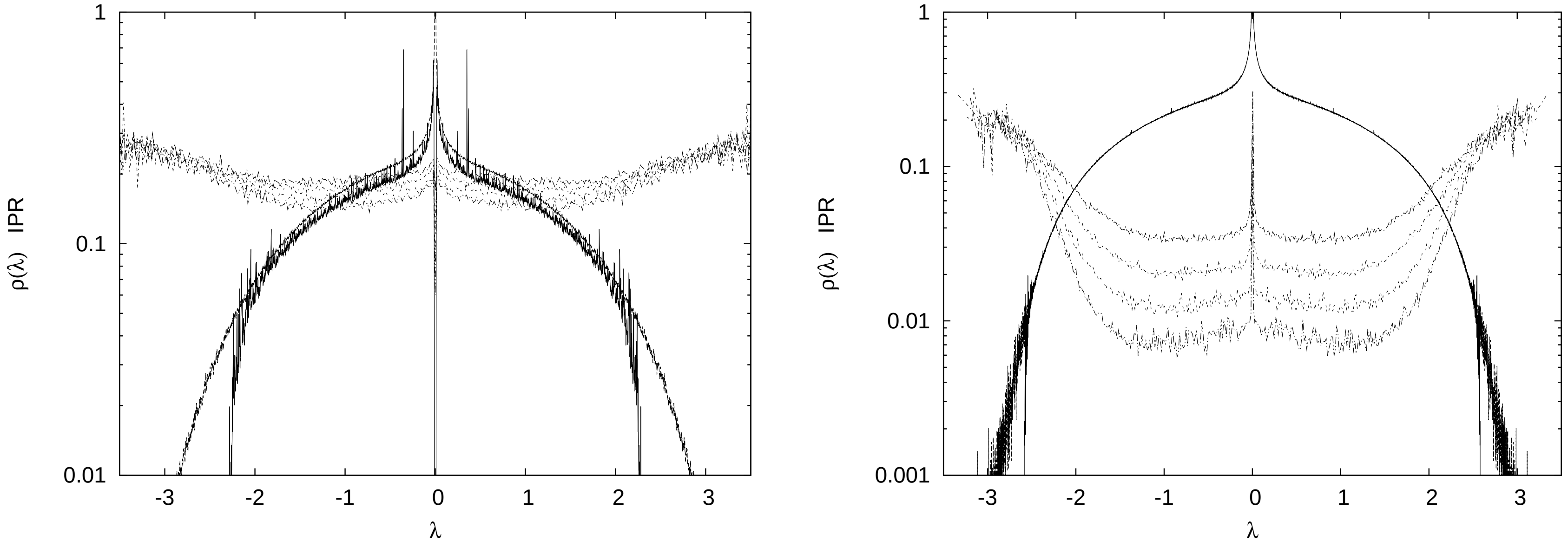


Upper left: zoom into the central region; upper right: results on logarithmic scale; lower: results regularized at $\varepsilon = 10^{-3}$. In all panels: numerical diagonalization results for graphs of size $N = 2000$ (dashed). Localization for $|\lambda| > 2.295$!

Localization — IPRs

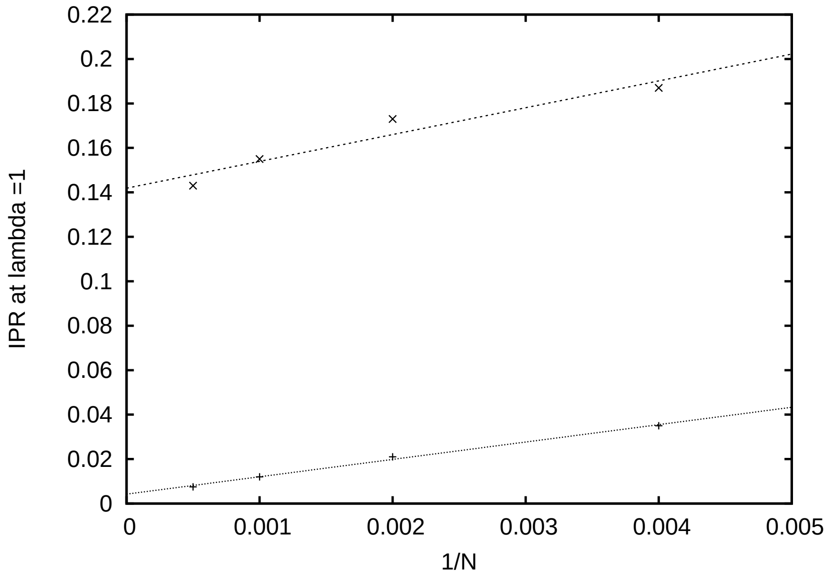
$$\mathrm{IPR}(\boldsymbol{v}) = \frac{\sum_{i=1}^{N} v_i^4}{\left(\sum_{i=1}^{N} v_i^2\right)^2}$$

- IPR $= \mathcal{O}(1)$ for localized, $\mathcal{O}(N^{-1})$ for de-localized states



Continuous and full densities of state, and average IPRs for Poissonian random graphs with $c = 2$ (left) and $c = 4$ (right). Average IPRs from numerical diagonalization of matrices with $N = 250$, $N = 500$, $N = 1000$ and $N = 2000$. Scaling of IPRs confirms location of mobility edges seen in DOS.

IPRs — Scaling with System Size



Scaling of average IPRs with system size for Poisson Random graphs with $c = 2$ (upper) and $c = 2$ (lower). The fraction of sites not in the giant cluster is $x_i \simeq 0.205$ at $c = 2$ and $x_i \simeq 0.02$ at $c = 4$.

Orther Ensembles

- Poisson random graphs with bimodal couplings

- Regular random graphs with Gaussian or bimodal couplings
(recover Wigner semi-circle law in the $c \gg 1$ limit)

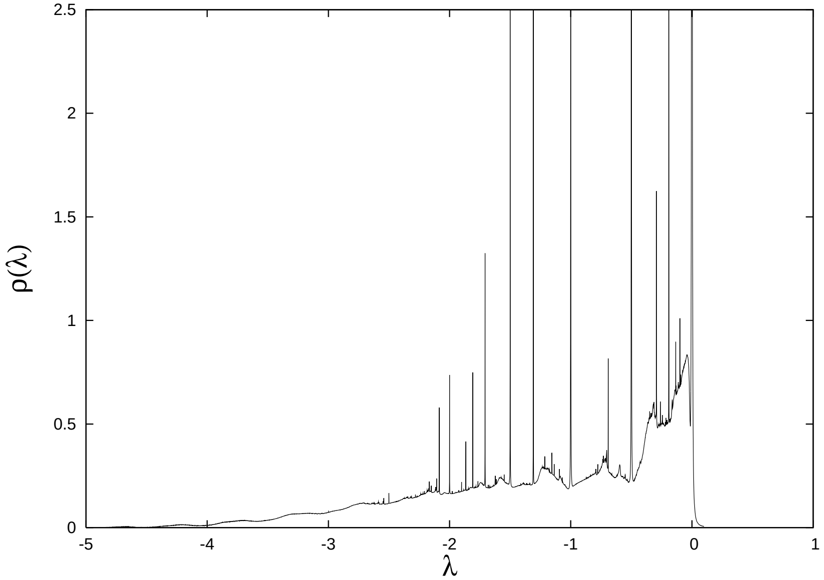
- Scale free graphs (power law degree distribution)
— For $p(k) = P_0 k^{-\gamma}$ confirm $\overline{\rho(\lambda)} \sim \lambda^{1-2\gamma}$ at large λ.

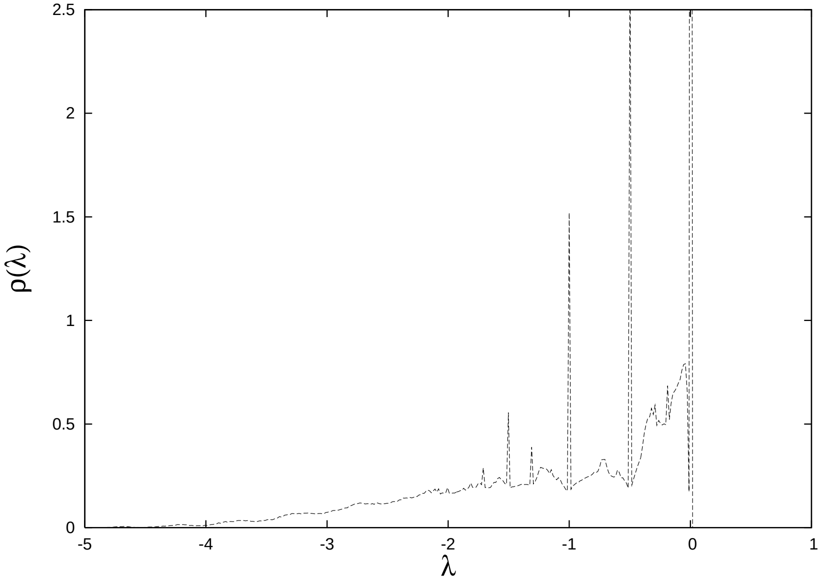
- In all cases: localization & mobility edges.

Results — Graph-Laplacians

- Spectra of matrices with row-constraints

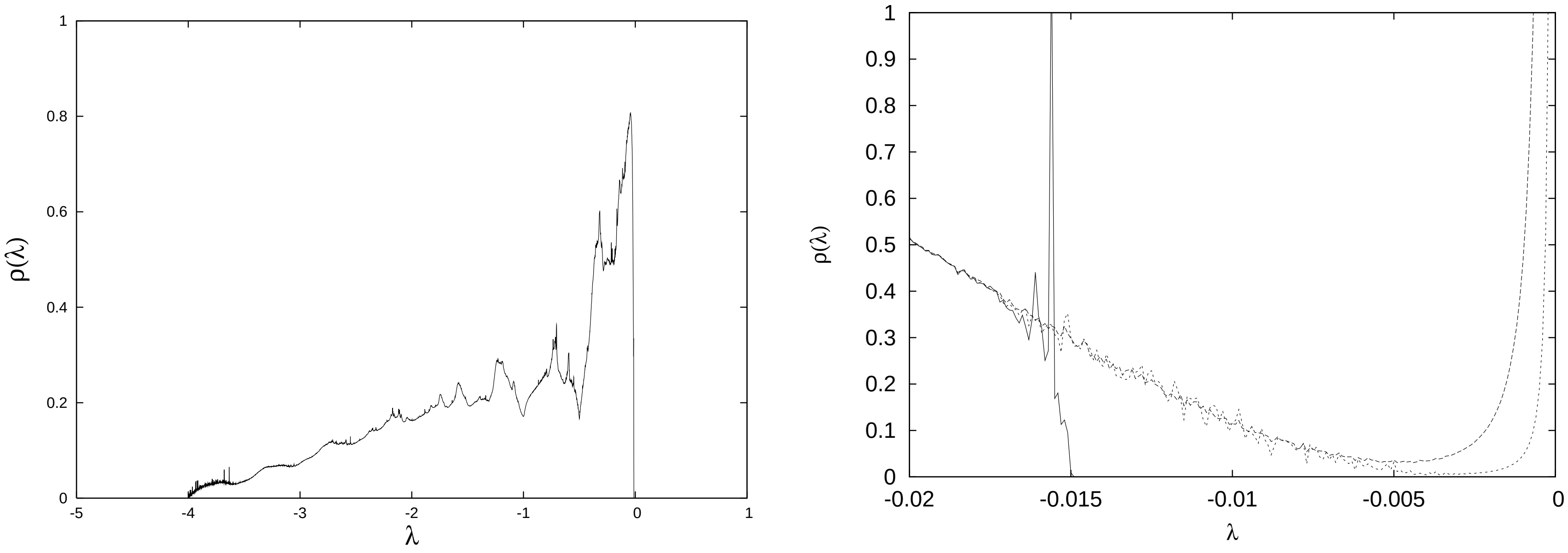
$$M_{ij} = c_{ij}K_{ij} - \delta_{ij}\sum_k c_{ik}K_{ik} \;; \qquad K_{ij} = 1/c \;\Leftrightarrow\; M = \Delta$$





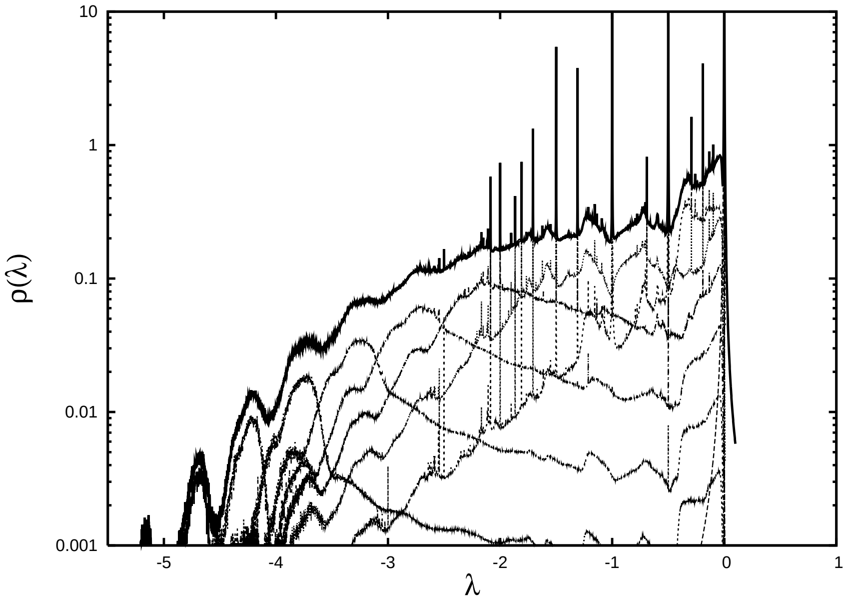
Spectral density for the Laplacian on a Poissonian random graph with $c = 2$ as computed via the present algorithm. Left: $\varepsilon = 10^{-3}$-results; right: results from numerical diagonalisation of $N \times N$ matrices of the same type with $N = 2000$.

Continuous Spectrum and 'Low-Energy' Lifshithz Tail



Spectral density for the Laplacian on a Poissonian random graph with $c = 2$. Left: continuous part of the spectrum obtained using $\varepsilon = 10^{-300}$ as a regularizer. Right: zoom into the small $|\lambda|$ region, exhibiting a mobility edge and a localized DOS ($\varepsilon = 10^{-5}$ and 10^{-6}) compatible with Lifshitz tail behaviour.

Results — Unfolding Spectral Densities

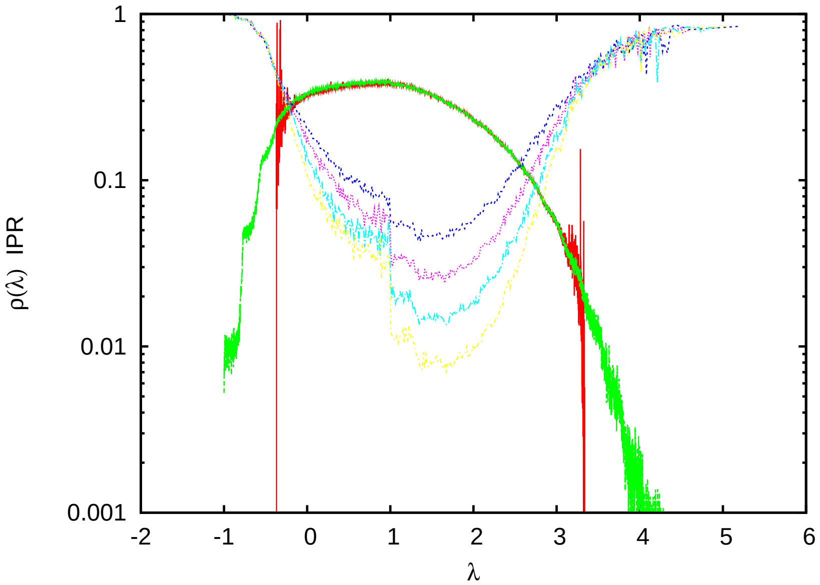


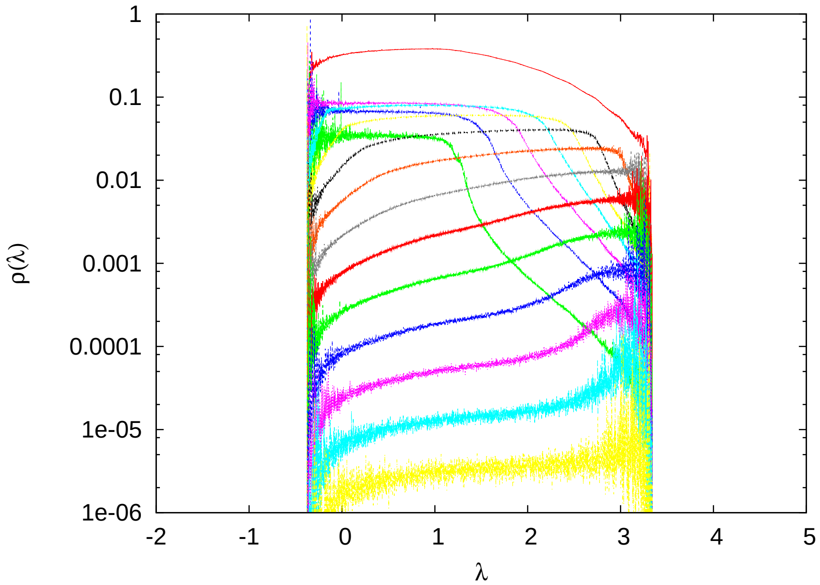
Spectral density for the Laplacian on a Poissonian random graph with $c = 2$ (full upper line), shown together with its unfolding according to contributions of different coordination. Identifiable humps from left tor right: $k = 9$, $k = 8$, $\ldots k = 3$. Several notable humps from $k = 2$, together with the $k = 1$ contribution mainly responsible for dip at $\lambda = -1$. The $k = 0$ contribution is mainly responsible for the δ-peak at $\lambda = 0$

Results — Random Schrödinger Operators

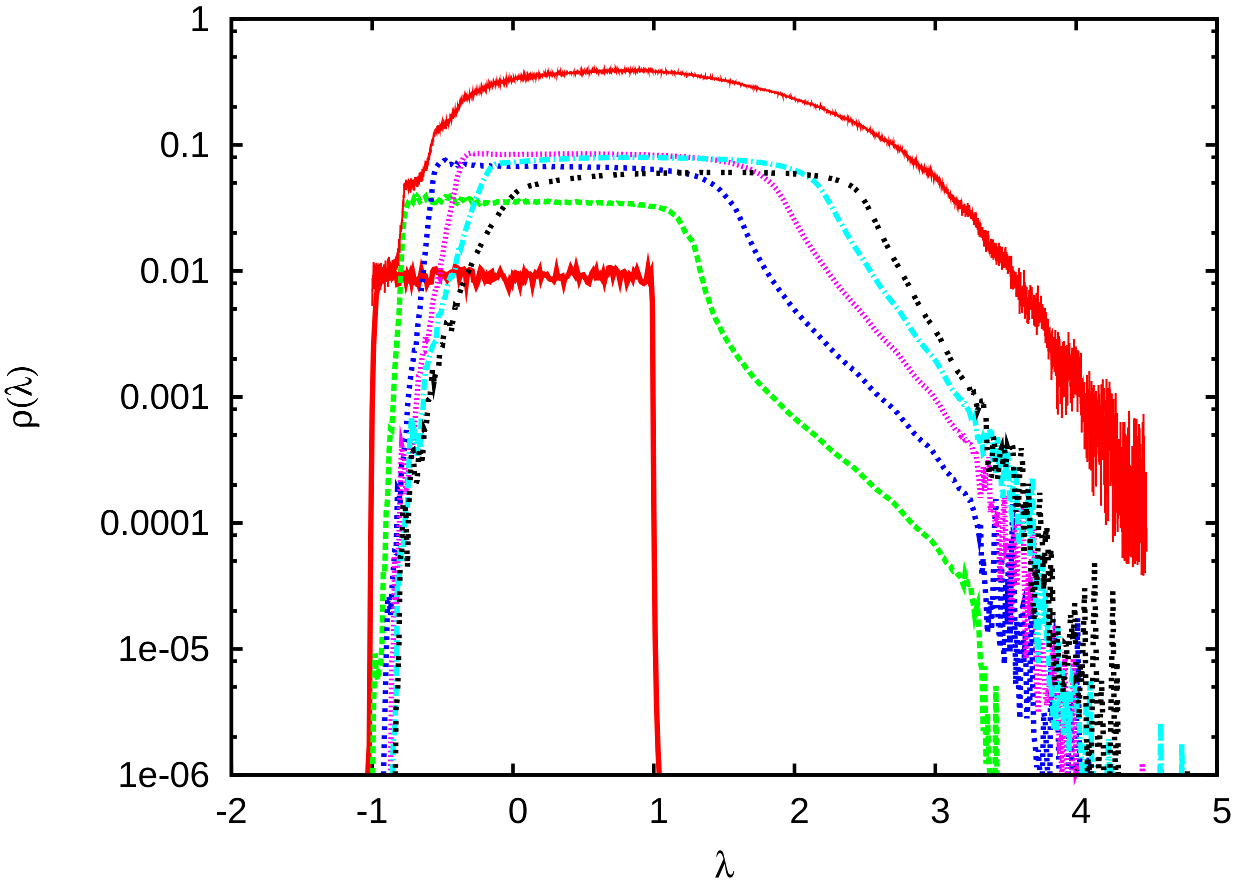
- Spectral properties of discrete random Schrödinger operator

$$H = -\Delta + V \ , \qquad V_{ij} = v_i \delta_{ij}, \qquad v_i \in [-W, W]$$





RSO on a Poissonian random graph with $c = 4$, and $W = 1$. Left: Spectral density and IPRs ($N = 250$, 500, 1000, and 2000. Right: Continuous DOS and its unfolding ($k = 1, \ldots 13$)



RSO on a Poissonian random graph with $c = 4$, and $W = 1$. DOS and its unfolding ($k = 0, \ldots 5$).

Summary

- Computed DOS of sparse random matrices using replica. For single instances see T. Rogers, et al PRE (2008).
- Techniques, ansätze etc inspired by previous work on Stat Mech of heterogeneous systems.
- Allows to disentangle pure point and continuous spectrum.
- Allows to compute local DOS unfolded according to coordination.
- Method is versatile (Poissonian and other degree distributions); Laplacians; discrete random Schrödinger operators; Anderson localisation.
- To do: asymmetric matrices (Rogers, Anand); modular& small world systems; eigenvector distributions, spectral correlations . . .