Fredholm determinant with the confluent hypergeometric kernel

J. Vasylevska
joint work with I. Krasovsky

Brunel University

Dec 19, 2008

Problem Statement

Let K_s be the integral operator acting on $L^2(-s,s)$ with confluent hypergeometric kernel:

$$K_s(u,v) = \frac{1}{2\pi}\frac{\Gamma(1-\beta+\alpha)\Gamma(1+\beta+\alpha)}{\Gamma(1+2\alpha)\Gamma(2+2\alpha)}\frac{A(u)B(v)-A(v)B(u)}{u-v},$$

$$A(x) = 2e^{-\pi i\beta sgn(x)/2}x|2x|^{\alpha}e^{-ix}\phi(\alpha+\beta+1, 2\alpha+2, 2ix),$$

$$B(x) = e^{-\pi i\beta sgn(x)/2}|2x|^{\alpha}e^{-ix}\phi(\alpha+\beta, 2\alpha, 2ix),$$

$\phi(a,c,z)$ is the confluent hypergeometric function;
α, β are complex parameters, $\Re\alpha > -1/2$.
Our aim is to obtain the asymptotics of the Fredholm determinant $P_s = \det(I-K_s)$ for large s.

- The kernel K_s is a particular case of a kernel that appears in the representation theory of the infinite-dimensional unitary group considered by Borodin and Olshanski.
- K_s can be obtained as a scaling limit at a Fisher-Hartwig singularity for a unitary random matrix ensemble. (i.e. a root-type singularity and a step-function)
- if $\beta = 0$ K_s reduces to the kernel in terms of Bessel functions:

$$K_s(u,v) = \frac{\sqrt{uv}}{2} \frac{J_{\alpha+\frac{1}{2}}(u)J_{\alpha-\frac{1}{2}}(v) - J_{\alpha+\frac{1}{2}}(v)J_{\alpha-\frac{1}{2}}(u)}{u-v},$$

  that was considered by Nagao, Slevin, Akemann et all, Witte, Forrester, Kuijlaars, Vanlessen. It can be obtained as a scaling limit at a root-type singular point for a unitary random matrix ensemble.
- if $\alpha = 0, \beta = 0$ K_s becomes the sine-kernel:

$$K_{\sin}(x,y) = \frac{\sin(x-y)}{\pi(x-y)}$$

Theorem (1)

The asymptotics for the Fredholm determinant $P_s = \det(I - K_s)$ on $(-s, s)$ as $s \to \infty$ are given by the formula

$$\ln P_s = -\frac{s^2}{2} + 2\alpha s - \left(\frac{1}{4} + \alpha^2 - \beta^2\right) \ln s - 2\alpha^2 \ln 2$$

$$-\ln \frac{G(1+\alpha+\beta)G(1+\alpha-\beta)}{G(1+2\alpha)} + c_0 + O\left(\frac{1}{s}\right), \text{ where}$$

$$c_0 = \frac{1}{12} \ln 2 + 3\zeta'(-1) = 2 \ln G(1/2) + \frac{1}{2} \ln \pi.$$

Theorem (1)

The asymptotics for the Fredholm determinant $P_s = \det(I - K_s)$ *on* $(-s, s)$ *as* $s \to \infty$ *are given by the formula*

$$\ln P_s = -\frac{s^2}{2} + 2\alpha s - \left(\frac{1}{4} + \alpha^2 - \beta^2\right) \ln s - 2\alpha^2 \ln 2$$

$$- \ln \frac{G(1+\alpha+\beta)G(1+\alpha-\beta)}{G(1+2\alpha)} + c_0 + O\left(\frac{1}{s}\right), \quad \textit{where}$$

$$c_0 = \frac{1}{12} \ln 2 + 3\zeta'(-1) = 2 \ln G(1/2) + \frac{1}{2} \ln \pi.$$

comparison with the sine-kernel asymptotics:

$$\ln \det(I - K_{\sin}) = -\frac{s^2}{2} - \frac{1}{4} \ln s + c_0 + O\left(\frac{1}{s}\right), \quad s \to \infty,$$

(des Cloizeaux and Mehta (1973), Dyson (1976), Widom (1994), Deift, Its, Zhou (1997), Krasovsky (2004), Ehrhardt(2007))

Two usual types of endpoints

- the density of eigenvalues vanishes as a square root ("soft edge" of the spectrum, e.g., GUE endpoints of semicircle). In the scaling limit at the endpoint one obtains the Airy kernel:

$$K_{Airy}(x,y) = \frac{\mathrm{Ai}\,(x)\mathrm{Ai}\,'(y) - \mathrm{Ai}\,(y)\mathrm{Ai}\,'(x)}{x-y}, \qquad \text{on } (-s,\infty)$$

Two usual types of endpoints

- the density of eigenvalues vanishes as a square root ("soft edge" of the spectrum, e.g., GUE endpoints of semicircle). In the scaling limit at the endpoint one obtains the Airy kernel:

$$K_{Airy}(x,y) = \frac{\mathrm{Ai}\,(x)\mathrm{Ai}\,'(y) - \mathrm{Ai}\,(y)\mathrm{Ai}\,'(x)}{x-y}, \qquad \text{on } (-s,\infty)$$

Asymptotics of the Tracy-Widom distribution:

$$\ln\det(I - K_{Airy}) = -\frac{s^3}{12} - \frac{1}{8}\ln s + \varkappa + O(s^{-3/2}), \quad s \to +\infty,$$

$$\varkappa = \frac{1}{24}\ln 2 + \zeta'(-1),$$

(Tracy, Widom (1994), Deift, Its, Krasovsky (2008), Baik, Buckingham, DiFranco (2008))

- the density of eigenvalues diverges as a square root ("hard edge" of the spectrum, e.g. the Laguerre ensemble at 0 or Jacobi ensemble at the edgepoints). In the scaling limit at the endpoint one obtains the Bessel kernel:

$$K_{Bes}(x,y) = \frac{\sqrt{x}J_{a+1}(\sqrt{x})J_a(\sqrt{y}) - \sqrt{y}J_a(\sqrt{x})J_{a+1}(\sqrt{y})}{2(x-y)}, \quad (a > -1),$$

on $(0,s)$. (Tracy, Widom (1994))

- the density of eigenvalues diverges as a square root ("hard edge" of the spectrum, e.g. the Laguerre ensemble at 0 or Jacobi ensemble at the edgepoints). In the scaling limit at the endpoint one obtains the Bessel kernel:

$$K_{Bes}(x,y) = \frac{\sqrt{x}J_{a+1}(\sqrt{x})J_a(\sqrt{y}) - \sqrt{y}J_a(\sqrt{x})J_{a+1}(\sqrt{y})}{2(x-y)}, \quad (a > -1),$$

on $(0,s)$. (Tracy, Widom (1994))

Theorem (2)

The large s asymptotics of $P_s = \det(I - K_{Bes})$ are given by

$$\ln\det(I - K_{Bes}) = -\frac{s}{4} + a\sqrt{s} - \frac{a^2}{4}\ln s + c_1 + O\left(\frac{1}{\sqrt{s}}\right),$$

where

$$c_1 = \ln\frac{G(1+a)}{(2\pi)^{a/2}}.$$

The main idea is to use a double-scaling limit of a Toeplitz determinant to obtain asymptotics of the Fredholm determinant P_s. The Toeplitz determinant with symbol f is given by the expression:

$$D_n(f) = \det\left(\frac{1}{2\pi}\int_0^{2\pi} e^{-i(j-k)\theta} f(\theta)d\theta\right)_{j,k=0}^{n-1}.$$

The main idea is to use a double-scaling limit of a Toeplitz determinant to obtain asymptotics of the Fredholm determinant P_s. The Toeplitz determinant with symbol f is given by the expression:

$$D_n(f) = \det\left(\frac{1}{2\pi}\int_0^{2\pi} e^{-i(j-k)\theta} f(\theta)d\theta\right)_{j,k=0}^{n-1}.$$

Let

$$f_\varphi(\theta) = \begin{cases} f(z) = |z-1|^{2\alpha} z^\beta e^{-i\pi\beta}, z = e^{i\theta}, & \varphi < \theta < 2\pi - \varphi, \\ 0, & otherwise, \end{cases}$$

The main idea is to use a double-scaling limit of a Toeplitz determinant to obtain asymptotics of the Fredholm determinant P_s. The Toeplitz determinant with symbol f is given by the expression:

$$D_n(f) = \det\left(\frac{1}{2\pi}\int_0^{2\pi} e^{-i(j-k)\theta} f(\theta) d\theta\right)_{j,k=0}^{n-1}.$$

Let

$$f_\varphi(\theta) = \begin{cases} f(z) = |z-1|^{2\alpha} z^\beta e^{-i\pi\beta}, z = e^{i\theta}, & \varphi < \theta < 2\pi - \varphi, \\ 0, & otherwise, \end{cases}$$

Then

$$P_s = \lim_{n\to\infty} \frac{D_n\left(f_{\frac{2s}{n}}\right)}{D_n(f)}.$$

The main idea is to use a double-scaling limit of a Toeplitz determinant to obtain asymptotics of the Fredholm determinant P_s. The Toeplitz determinant with symbol f is given by the expression:

$$D_n(f) = \det\left(\frac{1}{2\pi}\int_0^{2\pi} e^{-i(j-k)\theta} f(\theta)d\theta\right)_{j,k=0}^{n-1}.$$

Let

$$f_\varphi(\theta) = \begin{cases} f(z) = |z-1|^{2\alpha} z^\beta e^{-i\pi\beta}, z = e^{i\theta}, & \varphi < \theta < 2\pi - \varphi, \\ 0, & otherwise, \end{cases}$$

Then

$$P_s = \lim_{n\to\infty} \frac{D_n\left(f_{\frac{2s}{n}}\right)}{D_n(f)}.$$

Here the denominator is known (the Toeplitz determinant with a Fisher-Hartwig singularity, Böttcher, Silbermann):

$$D_n(f) = n^{\alpha^2-\beta^2}\frac{G(1+\alpha+\beta)G(1+\alpha-\beta)}{G(1+2\alpha)}(1+o(1)), \quad n\to\infty.$$

A classical representation of a Toeplitz determinant:

$$D_n(f_\varphi) = \prod_{k=0}^{n-1} \varkappa_k^{-2},$$

Here $p_k(z) = \varkappa_k z^k + \ldots$ are polynomials, orthonormal on the unit circle w.r.t. $f_\varphi(\theta)$.

$$\frac{d}{d\varphi} \ln D_n(f_\varphi) = \text{ in terms of } p_{n-1}(e^{\pm i\varphi}), p_n(e^{\pm i\varphi}).$$

A classical representation of a Toeplitz determinant:

$$D_n(f_\varphi) = \prod_{k=0}^{n-1} \varkappa_k^{-2},$$

Here $p_k(z) = \varkappa_k z^k + \dots$ are polynomials, orthonormal on the unit circle w.r.t. $f_\varphi(\theta)$.

$$\frac{d}{d\varphi} \ln D_n(f_\varphi) = \text{ in terms of } p_{n-1}(e^{\pm i\varphi}), p_n(e^{\pm i\varphi}).$$

We find the asymptotics of $p_n(z)$ by solving the associated Riemann-Hilbert problem (RHP).

$$\frac{d}{d\varphi} \ln D_n(f_\varphi) = -\frac{1}{2} n^2 \tan\frac{\varphi}{2} - \frac{n}{2}\left(2\alpha \tan\frac{\varphi}{2} - \frac{2\alpha}{\cos\frac{\varphi}{2}}\right) - \frac{1}{8}\frac{\cos\varphi/2}{\sin\varphi/2}$$

$$+\frac{\beta^2 \cos\varphi/2}{2\sin\varphi/2} - \alpha^2\left(\tan\frac{\varphi}{2} - \frac{1}{\cos\varphi/2} + \frac{\cos\varphi/2}{2\sin\varphi/2}\right) + O\left(\frac{1}{n\sin^2(\varphi/2)}\right),$$

where the remainder term is uniform for $2s/n \le \varphi \le \pi - \varepsilon, \quad \varepsilon > 0$.

$D_n(f_\varphi)-?$ as $\varphi \to \pi$ from below.

$$D_n(f_\varphi) = \frac{1}{(2\pi)^n n!} \int_{\varphi}^{2\pi-\varphi} \cdots \int_{\varphi}^{2\pi-\varphi} \prod_{1\le j<k\le n} |e^{i\theta_j} - e^{i\theta_k}|^2 \prod_{j=1}^{n} f(e^{i\theta_j}) d\theta_j .$$

$D_n(f_\varphi)-?$ as $\varphi \to \pi$ from below.

$$D_n(f_\varphi) = \frac{1}{(2\pi)^n n!} \int_{\varphi}^{2\pi-\varphi} \cdots \int_{\varphi}^{2\pi-\varphi} \prod_{1\le j<k\le n} |e^{i\theta_j} - e^{i\theta_k}|^2 \prod_{j=1}^{n} f(e^{i\theta_j}) d\theta_j .$$

After a change of variables we obtain:

$$D_n(f_\varphi) = \frac{\varepsilon^{n^2} 2^{2\alpha n}}{(2\pi)^n} \left(A_n + O(\varepsilon^2)\right), \quad \varphi = \pi - \varepsilon, \quad \varepsilon \to 0,$$

where

$$A_n = \frac{1}{n!} \int_{-1}^{1} \cdots \int_{-1}^{1} \prod_{1\le i<j\le n} (z_i - z_j)^2 \prod_{j=1}^{n} dz_j \quad - \text{ Selberg integral}$$

The asymptotics of A_n as $n \to \infty$ are known (Widom). Then, for $\varphi = \pi - \varepsilon$

$$\ln D_n(f_\varphi) = n^2(\ln\varepsilon - \ln 2) + 2\alpha n \ln 2 - \frac{1}{4}\ln n + \frac{1}{12}\ln 2 + 3\zeta'(-1) + O_n(\varepsilon),$$

where $O_n(\varepsilon) \to 0$, as $\varepsilon \to 0$.

$D_n(f_\varphi)-?$ as $\varphi \to \pi$ from below.

$$D_n(f_\varphi) = \frac{1}{(2\pi)^n n!} \int_{\varphi}^{2\pi-\varphi} \cdots \int_{\varphi}^{2\pi-\varphi} \prod_{1\le j<k\le n} |e^{i\theta_j} - e^{i\theta_k}|^2 \prod_{j=1}^{n} f(e^{i\theta_j}) d\theta_j .$$

After a change of variables we obtain:

$$D_n(f_\varphi) = \frac{\varepsilon^{n^2} 2^{2\alpha n}}{(2\pi)^n} \left(A_n + O(\varepsilon^2)\right), \quad \varphi = \pi - \varepsilon, \quad \varepsilon \to 0,$$

where

$$A_n = \frac{1}{n!} \int_{-1}^{1} \cdots \int_{-1}^{1} \prod_{1\le i<j\le n} (z_i - z_j)^2 \prod_{j=1}^{n} dz_j \quad - \text{ Selberg integral}$$

The asymptotics of A_n as $n \to \infty$ are known (Widom). Then, for $\varphi = \pi - \varepsilon$

$$\ln D_n(f_\varphi) = n^2(\ln\varepsilon - \ln 2) + 2\alpha n \ln 2 - \frac{1}{4}\ln n + \frac{1}{12}\ln 2 + 3\zeta'(-1) + O_n(\varepsilon),$$

where $O_n(\varepsilon) \to 0$, as $\varepsilon \to 0$.

By integrating $\frac{d}{d\varphi}\ln D_n(f_\varphi)$ and using the expression for $D_n(f_\varphi)$ as $\varphi \to \pi$, we obtain for any $\frac{2s}{n} \le \varphi \le \pi - \varepsilon$:

$$\ln D_n(f_\varphi) = (n^2+2n\alpha+2\alpha^2)\ln\cos\frac{\varphi}{2} + 2(n\alpha+\alpha^2)\ln\frac{1+\sin\frac{\varphi}{2}}{\cos\frac{\varphi}{2}} - \frac{1}{4}\ln n$$

$$-\left(\frac{1}{4}-\beta^2+\alpha^2\right)\ln\sin\frac{\varphi}{2} - 2\alpha^2\ln 2 + \frac{1}{12}\ln 2 + 3\zeta'(-1) + O\left(\frac{1}{n\sin\frac{\varphi}{2}}\right).$$

Setting $\varphi = \frac{2s}{n}$, and using

$$P_s = \lim_{n\to\infty}\frac{D_n\left(f_{\frac{2s}{n}}\right)}{D_n(f)},$$

we obtain Theorem 1.

Bessel kernel can be obtained as a scaling limit at the endpoint for the polynomials orthogonal on the interval $[-1, 1]$ that are related to the polynomials orthogonal on the unit circle with Fisher-Hartwig weight for $\beta = 0$. Consider the Hankel determinant with symbol $\omega(x)$:

$$D_n^H(\omega) = \det \left(\int_{-1}^{1} x^{j+k} \omega(x) dx \right)_{j,k=0}^{n-1}.$$

Bessel kernel can be obtained as a scaling limit at the endpoint for the polynomials orthogonal on the interval $[-1,1]$ that are related to the polynomials orthogonal on the unit circle with Fisher-Hartwig weight for $\beta = 0$. Consider the Hankel determinant with symbol $\omega(x)$:

$$D_n^H(\omega) = \det \left(\int_{-1}^{1} x^{j+k} \omega(x) dx \right)_{j,k=0}^{n-1}.$$

The Fredholm determinant $P_s^{Bes} = \det(I - K_s^{Bes})$:

$$P_s^{Bes} = \lim_{n\to\infty} \frac{D_n^H(\omega_{\frac{2s}{n}})}{D_n^H(\omega)}, \qquad \omega_\varphi(x) = \frac{f_\varphi(e^{i\theta})}{|\sin(\theta)|}, \quad x = \cos\theta.$$

Bessel kernel can be obtained as a scaling limit at the endpoint for the polynomials orthogonal on the interval $[-1, 1]$ that are related to the polynomials orthogonal on the unit circle with Fisher-Hartwig weight for $\beta = 0$. Consider the Hankel determinant with symbol $\omega(x)$:

$$D_n^H(\omega) = \det \left(\int_{-1}^{1} x^{j+k} \omega(x) dx \right)_{j,k=0}^{n-1} .$$

The Fredholm determinant $P_s^{Bes} = \det(I - K_s^{Bes})$:

$$P_s^{Bes} = \lim_{n\to\infty} \frac{D_n^H(\omega_{\frac{2s}{n}})}{D_n^H(\omega)}, \qquad \omega_\varphi(x) = \frac{f_\varphi(e^{i\theta})}{|\sin(\theta)|}, \quad x = \cos\theta.$$

Connection formula between Toeplitz and Hankel determinants:

$$\left(D_n^H(\omega)\right)^2 = \frac{\pi^{2n}}{2^{2(n-1)^2}} \frac{(\varkappa_{2n} + p_{2n}(0))^2}{p_{2n}(1)p_{2n}(-1)} D_{2n}(f(z)), \qquad \text{(Deift, Its, Krasovsky)}$$

here p_n are polynomials orthonormal on the unit circle with the weight $f(z)$.

- G. Akemann, P.H. Damgaard, U. Magnea, and S. Nishigaki: Universality of random matrices in the microscopic limit and the Dirac operator spectrum, Nucl. Phys. B **487**, no. 3, (1997), 721738.
- A.Borodin and G. Olshanski: Infinite random matrices and ergodic measures, Comm. Math. Phys. **223** (2001). no.1, 87-123.
- J. des Cloizeaux and M. L. Mehta. Asymptotic behaviour of spacing distributions for the eigenvalues of random matrices, J. Math. Phys. **14**, 1648-1650 (1973)
- P. Deift, A. Its, and I. Krasovsky: Toeplitz and Hankel determinants with Fisher-Hartwig singularities
- P. Deift, A.Its, and I.Krasovsky: Asymptotics of the Airy-kernel determinant. Comm. Math. Phys. **278**, 643-678 (2008)
- P. Deift, A. Its, and X. Zhou. A Riemann-Hilbert approach to asymptotic problems arising in the theory of random matrix models, and also in the theory of integrable statistical mechanics. Ann. Math. **146**, 149-235 (1997)

F. Dyson: Fredholm determinants and inverse scattering problems. Commun. Math. Phys. **47**, 171-183 (1976)

T.Erhardt: Dyson's constant in the asymptotics of the Fredholm determinant of the sine kernel. Comm. Math. Phys. **272**, 683-698 (2007)

I.V.Krasovsky: Gap probability in the spectrum of random matrices and asymptotics of polynomials orthogonal on an arc of the unit circle. Int. Math. Res. Not. **2004**, 1249-1272 (2004)

T.Nagao, K.Slevin: Nonuniversal correlations of random matrix ensembles. J. Math. Phys., **34** (1993), pp.2075-2085.

C.A. Tracy, H. Widom: Level Spacing Distributions and the Bessel kernel. Comm. Math. Ph. **161**, 289-309 (1994)

H. Widom: The asymptotics of a continuous analogue of orthogonal polynomials. J. Approx. Th. **77**, 51-64 (1994)

N.S. Witte, P.J. Forrester, Gap probabilities in the finite and scaled Cauchy random matrix ensembles, Nonlinearity **13** (200), no. 6, 1965-1986.