V BRUNEL Workshop on Random Matrix Theory
(Brunel University – West London)

Horizon in Random Matrix Theory, Hawking Radiation and Flow of Cold Atoms

by

Fabio Franchini

Coauthor: V. E. Kravtsov

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Thanks: R. Balbinot, S. Fagnocchi & I. Carusotto
(also for the figures)

The star of the talk:

- Two-Point (Density-Density) correlation function:

$$Y_2^a(x,x') = \frac{\kappa^2}{4\pi^2} \frac{\sin^2[\pi(x-x')]}{\cosh^2[\kappa(x+x')/2]} , \qquad \text{for } \; x\,x' < 0$$

(Anomalous: non-translational invariant)

$$Y_2^n(x,x') = \frac{\kappa^2}{4\pi^2} \frac{\sin^2[\pi(x-x')]}{\sinh^2[\kappa(x-x')/2]} , \qquad \text{for } \; x\,x' > 0$$

(Normal: translational invariant)

Same correlator for different systems

Invariant RME with Weak Confinement

?

Acoustic Black Hole in a BEC

$$Y_2^a(x,x') \propto \cosh^{-2}\left[\frac{\kappa(x+x')}{2}\right]$$

$$\text{for} \quad x\, x' < 0$$

Luttinger Liquid in curved space-time

Outline

- RME with Weak Confinement
- Acoustic Black Hole in a BEC
- Hawking radiation
- Luttinger Liquid in curved metric & RME
- Conclusions

Invariant Ensembles

- Invariant Probability Distribution Function:

$$P(\mathbf{H}) \propto \mathrm{e}^{-\mathbf{Tr}V(\mathbf{H})}$$

- Describe extended states (no localization)

$\rightarrow$ Wigner statistics

- Gaussian Ensemble: $P(\mathbf{H}) \propto \mathrm{e}^{-\sum_{n,m} |H_{nm}|^2/\sigma^2}$

$$\langle H_{n,m} \rangle = 0 \,, \qquad \langle H_{n,m}^2 \rangle = \sigma^2$$

Non-Invariant Ensembles

- Non-Invariant PDF:

$$P(\mathbf{H}) \propto \mathrm{e}^{-\sum_{n,m} A_{nm} \, |H_{nm}|^2} \Rightarrow \langle H^2_{n,m} \rangle = A^{-1}_{nm}$$

- Localized states (Poisson statistics) $\rightarrow$ $A_{nm} = \mathrm{e}^{|n-m|/B}$

- Multi-Fractal states (Critical Statistics) $\rightarrow$ $A_{nm} = 1 + \frac{(n-m)^2}{B^2}$

- …

Weakly confined Invariant Ensemble

$$P(\mathbf{H}) \propto \mathrm{e}^{-\mathrm{Tr} V(\mathbf{H})} \,, \quad V(E) \overset{|E|\to\infty}{\simeq} \kappa \, \ln^2 |E|$$

- Critical Statistics
  (Spontaneous Breaking of Invariance?)

- Exactly solvable (q-deformed Hermite Polynomial):

(Muttalib et al. '93)

$$V(E) = \sum_{n=0}^{\infty} \ln\left[1 + 2q^{n+1}\cosh(2\chi) + q^{2n+2}\right]$$

$$E \equiv \sinh\chi, \;\; q \equiv \mathrm{e}^{-\kappa}$$

Weakly Confined Invariant Ensemble

$$P(\mathbf{H}) \propto \mathrm{e}^{-\mathrm{Tr}V(\mathbf{H})}\,, \quad V(E) \overset{|E|\to\infty}{\simeq} \kappa\, \ln^2 |E|$$

- Non-Trivial density eigenvalue distribution

- Unfolding to make density constant:

$$\rho(E) \equiv \mathrm{tr}\left\{\delta\left(E - \mathbf{H}\right)\right\}$$

$$\boxed{E_x = \lambda\, \mathrm{e}^{\kappa|x|}\, \mathrm{sign}(x)}$$

$$\langle\tilde{\rho}(x)\rangle \equiv \langle\rho(E_x)\rangle\, \frac{\mathrm{d}E_x}{\mathrm{d}x} = 1$$

Weakly Confined Invariant Ensemble

$$P(\mathbf{H}) \propto \mathrm{e}^{-\mathrm{Tr} V(\mathbf{H})}\,, \quad V(E) \stackrel{|E|\to\infty}{\simeq} \kappa \, \ln^2 |E|$$

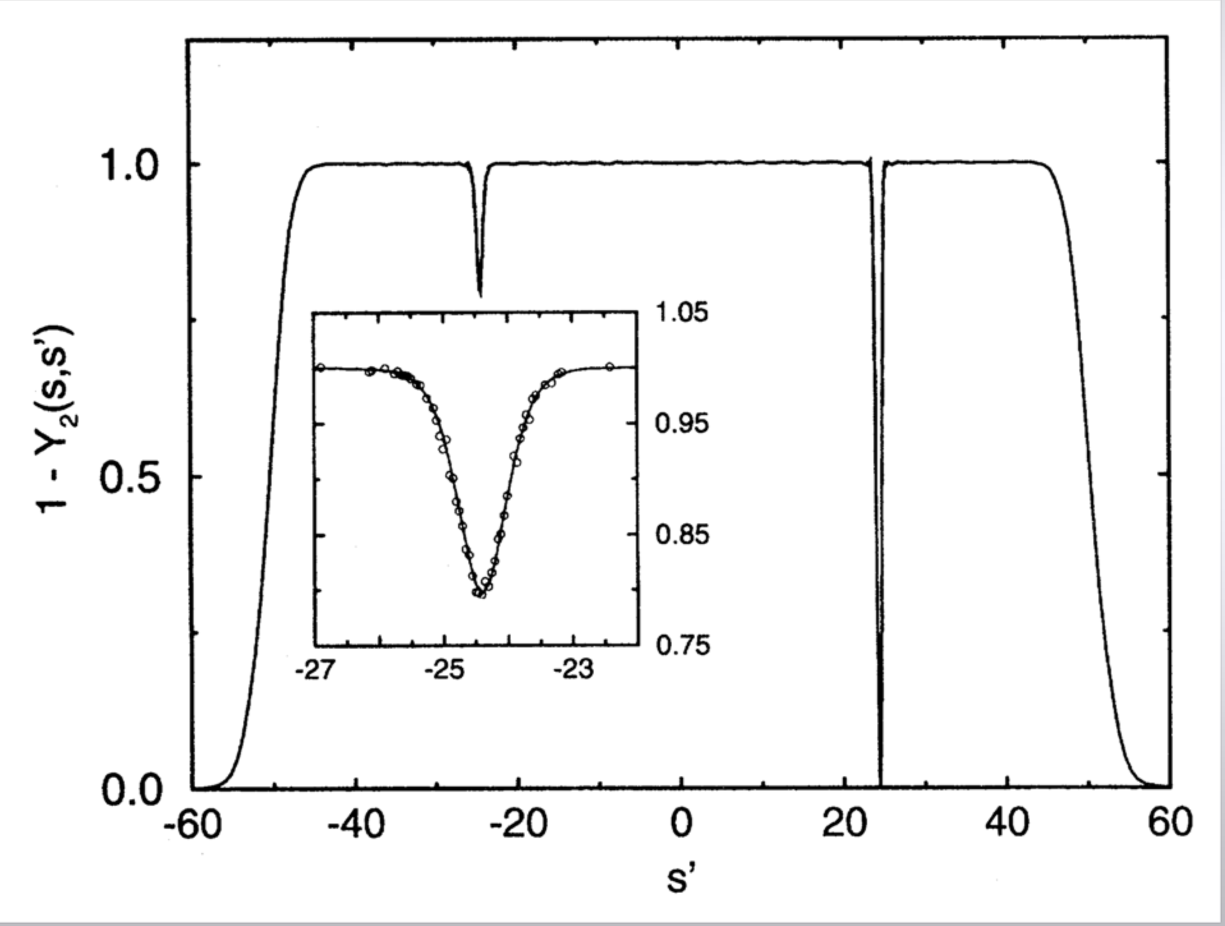
- For $e^{-2\pi^2/\kappa} << 1$ semiclassical analysis (Canali et al '95):

$$\begin{aligned} Y_2(x,x') \quad = \quad & \frac{\kappa^2}{4\pi^2} \frac{\sin^2[\pi(x-x')]}{\sinh^2[\kappa(x-x')/2]} \, \theta(x\,x') \\ & + \frac{\kappa^2}{4\pi^2} \frac{\sin^2[\pi(x-x')]}{\cosh^2[\kappa(x+x')/2]} \, \theta(-\,x\,x') \end{aligned}$$

$$\boxed{Y_2(x,x') \equiv \delta(x-x') - \frac{\langle \rho(E_x)\rho(E_{x'})\rangle - \langle \rho(E_x)\rangle\langle \rho(E_{x'})\rangle}{\langle \rho(E_x)\rangle\langle \rho(E_{x'})\rangle}}$$

Weakly Confined Invariant Ensemble

- Numerical check (Canali et al ’95):



First Interlude

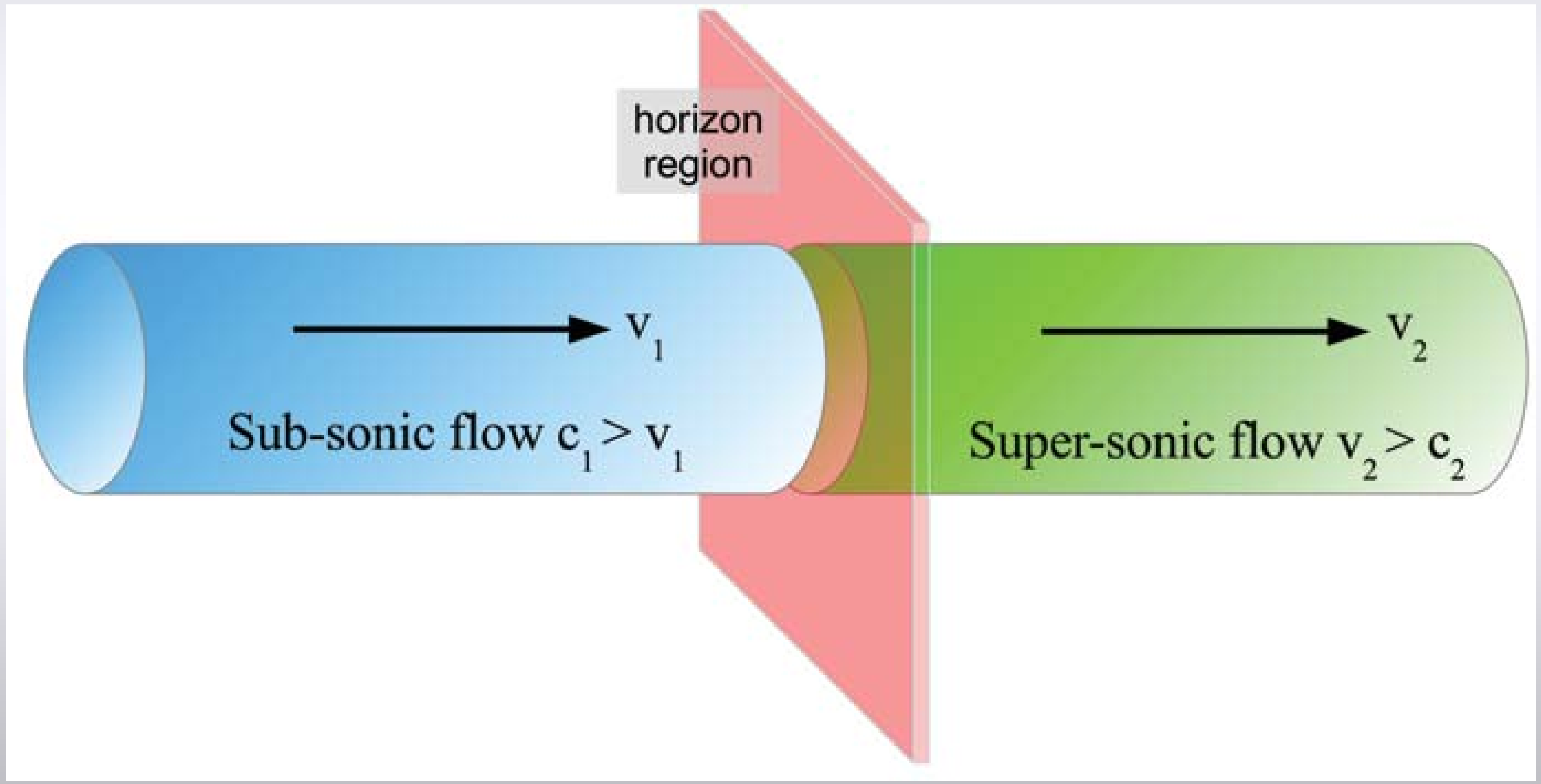
- Interesting RME with interesting correlator
- Mathematically exact $\rightarrow$ physical interpretation?

Let's abandon RMT for a moment

and look at something completely different…

Acoustic Black-Hole

- Fluid pushed to move faster than it's speed of sound:



Hawking radiation

- Prediction: a Black Hole radiates particles with an exact thermal (*Black-Body*) spectrum
- Solid result due only to horizon (kinematical)
- Different ways to understand it:
  - Pair production close to horizon
  - Red-shifting of last escaping modes
  - Casimir effect
  - Bogoliuobov overlap of positive frequency modes close to the horizon and at infinity
  - …

QFT in Curved Space-Time

- Field quantization is basis-dependent:

$$\phi(x) = \sum_i \left[a_i f_i(x) + a_i^\dagger f_i^*(x) \right]$$

Plane waves

$\Rightarrow$ vacuum depends on the observer: $a_i |0\rangle_x = 0 \ \forall i$

- For a different coordinate system:

$$\phi(\tilde{x}) = \sum_i \left[\tilde{a}_i \tilde{f}_i(\tilde{x}) + \tilde{a}_i^\dagger \tilde{f}_i^*(\tilde{x}) \right] \ , \quad \tilde{a}_i |\tilde{0}\rangle_{\tilde{x}} = 0 \ \forall i$$

$$\Rightarrow \quad \boxed{{}_x\langle 0|\tilde{0}\rangle_{\tilde{x}} \neq 1}$$

Hawking Radiation

- If $\tilde{x} \sim \mathrm{e}^{\kappa|x|}\mathrm{sgn}(x)$

$$\langle 0|\tilde{N}_\omega|0\rangle = \frac{1}{\mathrm{e}^{2\pi\omega/\kappa} - 1} = \frac{1}{\mathrm{e}^{\hbar\omega/k_B T_H} - 1}$$

$\Rightarrow$ black body radiation with temperature

$$T_H = \frac{\hbar\kappa}{2\pi k_B}$$

(pure quantum effect!)

Search for Hawking radiation

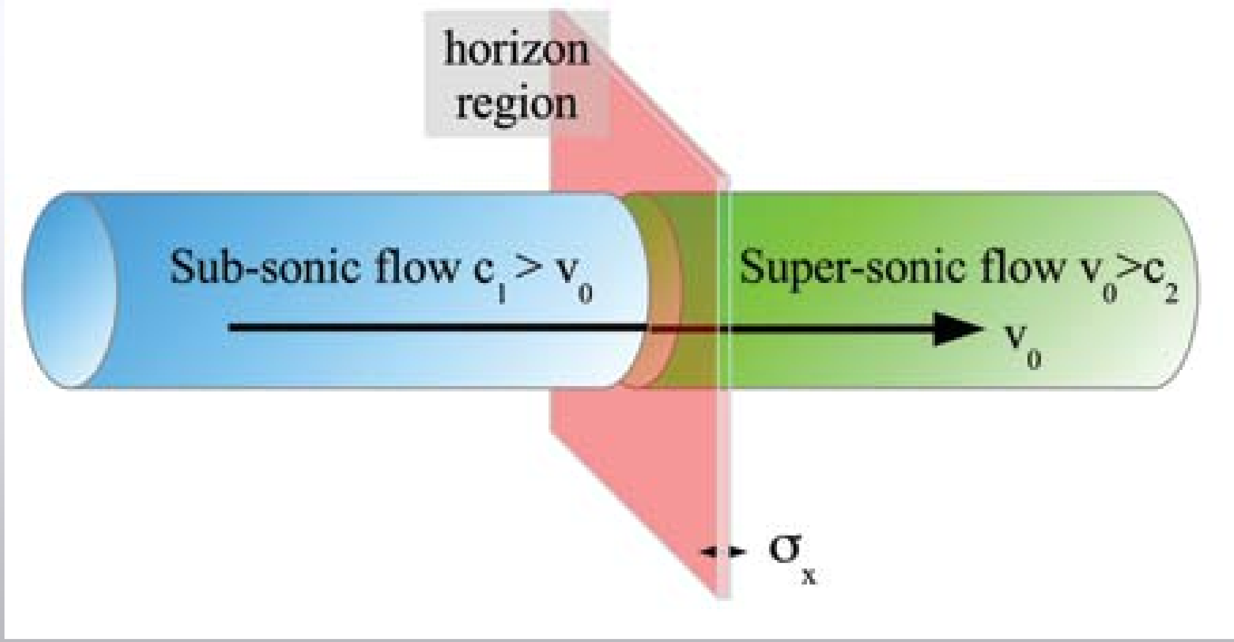
- Not yet observed (too small)
- Solid theoretical prediction: general phenomenon

$\Rightarrow$ Analogue Gravitational Models

- BEC: $T_H \approx 0.01 \div 0.1\ T_{BEC}$: still small for detection
- Search for different signature

Acoustic BH in BEC

- Cool system → Bose-Einstein Condensate
- Keep stream velocity constant & change speed of sound
- Effective dynamics is 1-D



BEC Phonons

- Bose field: $\Psi = \sqrt{\rho}\, e^{i\phi}$ $\begin{cases} \rho = \rho_0 + \delta\rho : & \textbf{number density} \\ \phi = \phi_0 + \delta\phi : & \textbf{phase} \end{cases}$

- ρ_0 & ϕ_0 by mean-field Gross-Pitaaevskii

- $\delta\phi$: sound wave over background fluid

- $\delta\rho$: algebraically related to $\delta\phi$

Effective Gravity in fluids

- Low-Energy excitations (phonons) propagate on top of bulk stream
- Effective dynamics: D'Alembert equation in curved metric ($c^2 = dp/d\rho$):

$$\sqrt{(-g)}g^{\mu\nu}\nabla_\mu\nabla_\nu\delta\phi = \partial_\mu\left[\sqrt{(-g)}g^{\mu\nu}\partial_\nu\delta\phi\right]$$

$$\sqrt{(-g)}g^{\mu\nu} = \rho_0\begin{pmatrix} \frac{1}{c^2} & \frac{v^i}{c^2} \\ \frac{v^j}{c^2} & \frac{v^i v^j}{c^2} - \delta^{ij} \end{pmatrix}$$

2-Point Correlator

- In flat space: $\langle \delta\phi(x,t)\delta\phi(x',t')\rangle \propto \ln\left(\Delta u^{+}\ \Delta u^{-}\right)$

$$u^{\pm} \equiv t \pm \int \frac{\mathrm{d}x}{c \mp v} \longleftarrow$$

Light-Cone coordinates

and for the density: $\langle \delta\rho(x,0)\delta\rho(x',0)\rangle \propto \dfrac{1}{(x-x')^2}$

- Around the black hole: $u^{-} \to \tilde{u}^{-} \equiv \dfrac{1}{\kappa} \mathrm{e}^{-\kappa u^{-}} \mathrm{sgn}(x)$

2-Point Correlator in curved metric

$$\langle \delta\rho(x,0)\delta\rho(x',0)\rangle \propto \cosh^{-2}\left[\frac{\kappa}{2}\left(\frac{x}{c_r - v} + \frac{x'}{v - c_l}\right)\right]$$

(Balbinot et al. '08)

$$\text{for} \quad x\, x' < 0$$

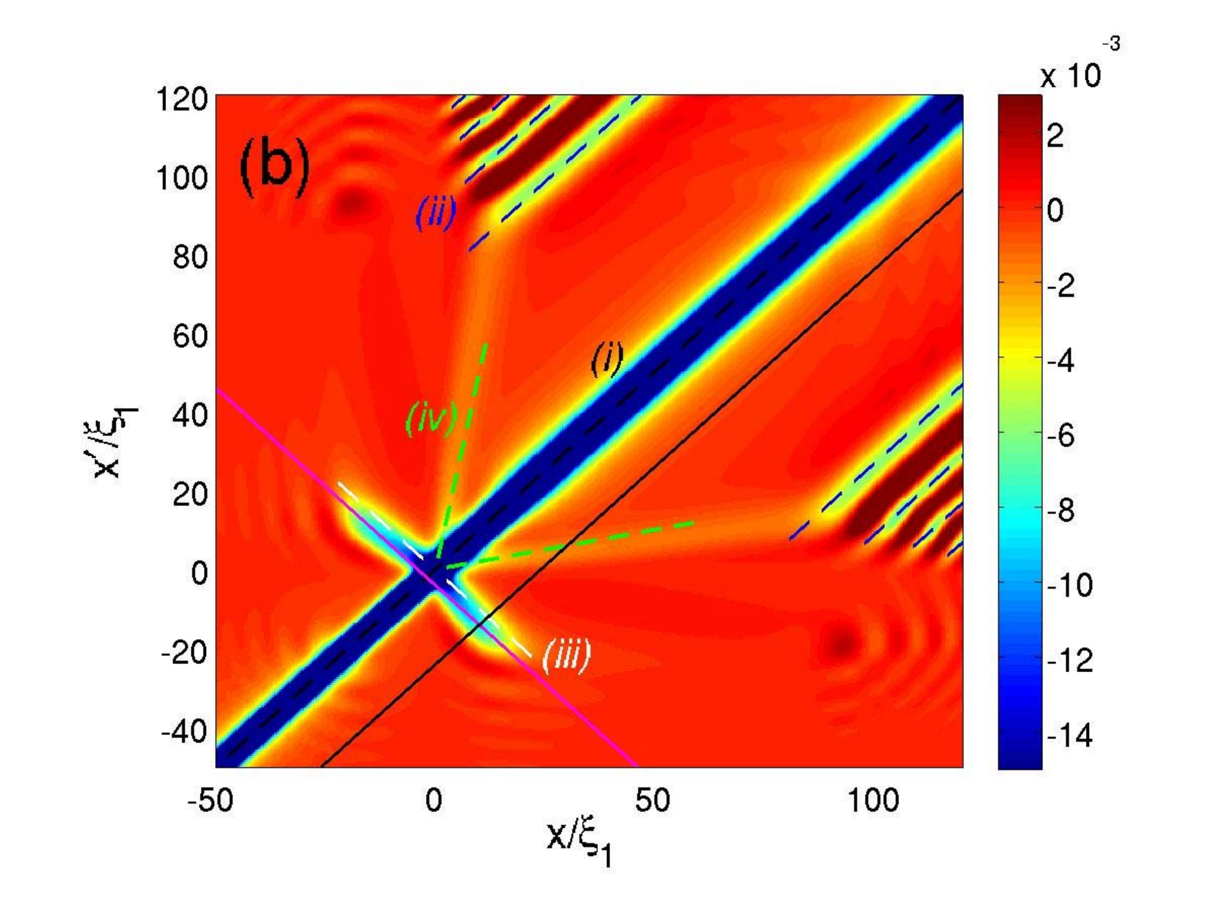
- Same non-local correlation as RME for $c_{r,l} = v \pm v/2$

$$Y_2^a(x,x') = \frac{\kappa^2}{4\pi^2} \frac{\sin^2\left[\pi(x - x')\right]}{\cosh^2\left[\kappa(x + x')/2\right]}$$

(except for the oscillatory term...)

Numerical check

- Field theory prediction checked against ab-initio numerical simulation (Carusotto et al. ’08)



Second Interlude

- BEC system has non-local signature
- Low-Energy description in terms of free field in curved metric with horizon

Let's go back to RMT

and apply what we have learned

Effective Theory for RME

$$\mathcal{L} = -\beta \sum_{n>m} \ln |E_n - E_m| + \sum_n V(E_n)$$

- Energy eigenvalues
  $\rightarrow$ coordinates of interacting particles
  (fermions $\Leftarrow$ level repulsion)

- Parametric evolution of RME
  $\rightarrow$ time coordinate

- Eigenvalue distribution
  $\rightarrow$ ground state configuration of 1D quantum model

Effective Theory for RME

- Low-Energy effective theory for 1-D system:

Luttinger Liquid
$$
\begin{cases}
\Psi(x,\tau) \simeq \Psi_R\, \mathrm{e}^{\mathrm{i}k_F x} + \Psi_L\, \mathrm{e}^{-\mathrm{i}k_F x} \\
\Psi_{R,L} \propto \mathrm{e}^{\pm \mathrm{i}\Phi_{R,L}(x,\tau)}
\end{cases}
$$

- Low-Energy effective theory for 1-D system:

$$
\mathcal{S}[\Phi] = \frac{1}{2\pi K} \int \mathrm{d}\tau \int \mathrm{d}x \left[\frac{1}{c} \left(\partial_\tau \Phi\right)^2 + c \left(\partial_x \Phi\right)^2 \right]
$$

Luttinger theory for RME

$$\rho(x,\tau) = \rho_0 - \frac{1}{\pi}\,\partial_x\Phi + \frac{A_K}{\pi}\cos\left[2\pi\rho_0 x - 2\Phi\right] + \ldots$$

Unfolding:

$\rho_0 = 1$

- Two-Point function (Kravtsov et al. '00):

$$\begin{aligned} Y_2 &= -\frac{1}{\pi^2}\langle\partial_x\Phi(x)\partial_{x'}\Phi(x')\rangle \\ &\quad -\frac{A_K^2}{2\pi^2}\cos(2\pi(x-x'))\langle \mathrm{e}^{\mathrm{i}2\Phi(x)}\mathrm{e}^{-\mathrm{i}2\Phi(x')}\rangle + \ldots \end{aligned}$$

- In flat space: $\langle\Phi(x,t)\Phi(x',t')\rangle \propto \ln\left(\Delta x^2 + \Delta t^2\right)$

$$Y_2 \propto \frac{\sin^2\left[\pi(x-x')\right]}{(x-x')^2}$$

2-Point Function for Gaussian RME

(K=1: Unitary)

Non-invariant Critical Ensemble

- Critical Random Banded Matrix (Multifractal spectrum)

$$P(\mathbf{H}) \propto \mathrm{e}^{-\sum_{n,m} A_{nm} \, |H_{nm}|^2} \qquad A_{nm} = 1 + \frac{(n-m)^2}{B^2}$$

- Thermal effective Luttinger Theory (Kravtsov & Tsvelik-2001)

$g*$: critical conductance

$$\mathcal{S}[\Phi] = \frac{1}{2\pi K} \int^{g*} \mathrm{d}\tau \int \mathrm{d}x \, \left[\frac{1}{c} \left(\partial_\tau \Phi\right)^2 + c \left(\partial_x \Phi\right)^2 \right]$$

$$Y_2(x, x') = T^2 \, \frac{\sin^2 [\pi(x - x')]}{\sinh^2 [\pi T (x - x')]}, \; T \equiv \frac{1}{g*}$$

Thermal Field Theory

- Diagonal part of 2-point function

$$Y_2(x,x') = T^2 \, \frac{\sin^2\left[\pi(x-x')\right]}{\sinh^2\left[\pi T(x-x')\right]}$$

common to

- weakly confined invariant ensemble
- Lorentzian banded matrix ensembles
- Standard thermal field theory

- How to generate the non-translational invariant part?

Luttinger theory in curved metric

$$P(\mathbf{H}) \propto \mathrm{e}^{-\mathrm{Tr} V(\mathbf{H})} \, , \; V(E) \overset{|E|\to\infty}{\simeq} \kappa \, \ln^2 |E|$$

- BEC system taught us that metric with horizons gives non-local correlation function

$$\mathcal{S}[\Phi] = \frac{1}{8\pi K} \int \mathrm{d}^2\xi \sqrt{g(\xi)} g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi$$

- In 1+1 D any horizon metric can be approximated by

$$g \equiv \det g^{\mu\nu} \, , \; \mathrm{d}s^2 = g_{\mu\nu} \mathrm{d}\xi^\mu \mathrm{d}\xi^\nu$$

Rindler line element

$$\mathrm{d}s^2 = -y^2 \, \mathrm{d}t^2 + \frac{1}{\kappa^2} \, \mathrm{d}y^2$$

Horizon at $y=0$

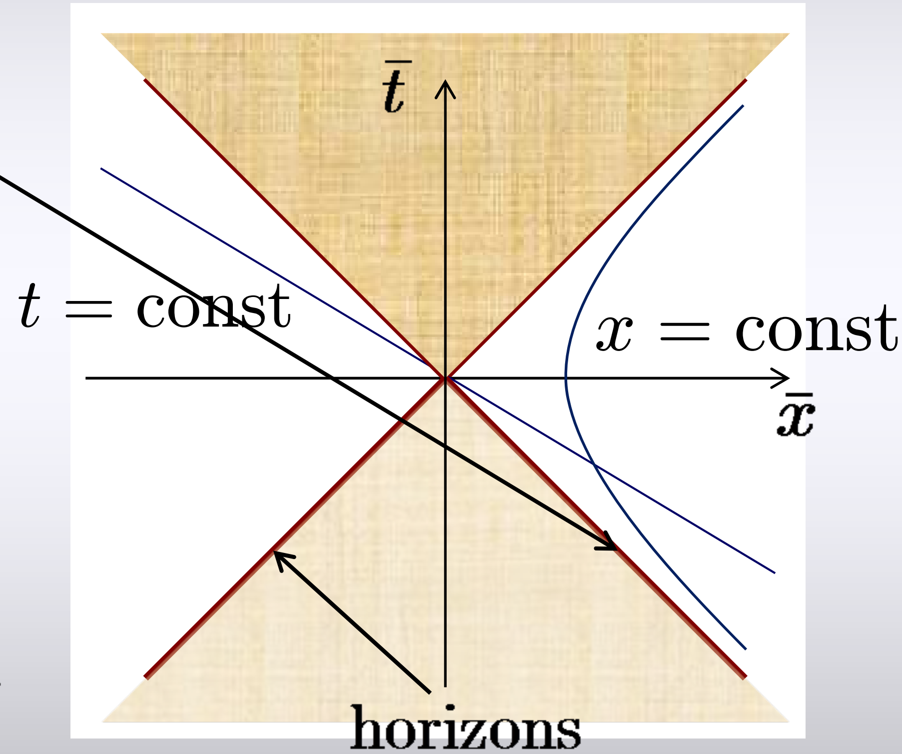
- Let's choose: $y \equiv \sinh(\kappa x)$

Luttinger theory in Rindler space

$$\begin{cases} \bar{t} & \equiv & \dfrac{1}{\kappa} \sinh \kappa x \ \sinh \kappa t \\ \bar{x} & \equiv & \dfrac{1}{\kappa} \sinh \kappa x \ \cosh \kappa t \end{cases}$$

Periodic in imaginary time
$\rightarrow$ finite **temperature**

$$\begin{aligned} \mathrm{d}s^2 &= -\sinh^2(\kappa x)\, \mathrm{d}u^+\, \mathrm{d}u^- \\ &= -\,\mathrm{d}\bar{u}^+\, \mathrm{d}\bar{u}^- \\ \bar{u}^\pm &\equiv \bar{t} \pm \bar{x} \end{aligned}$$



- Far from the origin:

$$\bar{u}^\pm \simeq \begin{cases} \pm \dfrac{\mathrm{e}^{\pm\kappa u^\pm}}{2\kappa}, & x \gg 1 \\ \mp \dfrac{\mathrm{e}^{\pm\kappa u^\mp}}{2\kappa}, & x \ll -1 \end{cases}$$

Luttinger Liquid in Rindler Space

- Remind two-Point function:

$$Y_2 = -\frac{1}{\pi^2}\langle \partial_x \Phi(x) \partial_{x'} \Phi(x') \rangle - \frac{A_K^2}{2\pi^2} \cos(2\pi(x - x')) \langle e^{i2\Phi(x)} e^{-i2\Phi(x')} \rangle + \dots$$

- With the new coordinates: $\left(\bar{x} = \frac{e^{\kappa|x|}}{2\kappa} \operatorname{sgn}(x) \right)$

$$\langle \Phi(x)\Phi(x') \rangle \overset{|x|,|x'| \gg 1}{\propto} \begin{cases} \ln\left[\frac{2}{\kappa} \sinh \frac{\kappa(x-x')}{2}\right], & x\, x' > 0 \\ \ln\left[\frac{2}{\kappa} \cosh \frac{\kappa(x+x')}{2}\right], & x\, x' < 0 \end{cases}$$

Luttinger Liquid in Rindler Space

- We recover exactly the RME correlation (K=1):

$$Y_2^a(x,x') = \frac{\kappa^2}{4\pi^2} \frac{\sin^2[\pi(x-x')]}{\cosh^2[\kappa(x+x')/2]} , \qquad \text{for} \quad x\, x' < 0$$

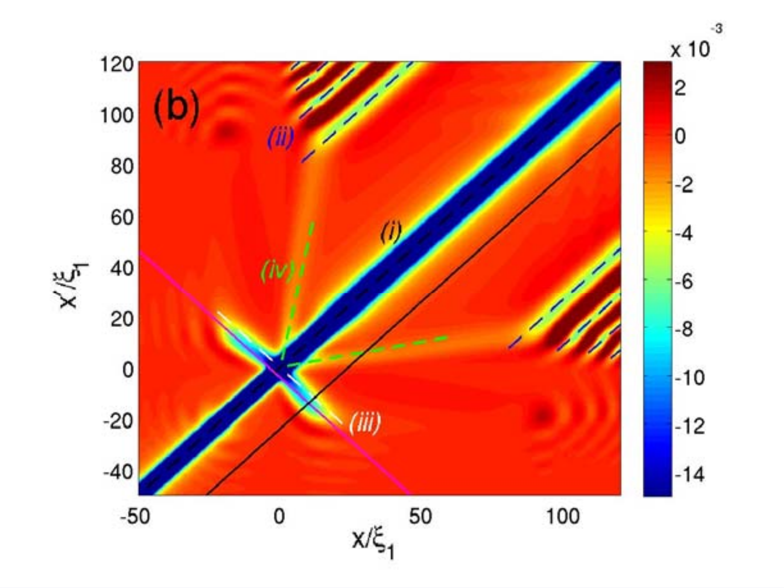
(Anomalous: non-translational invariant)

$$Y_2^n(x,x') = \frac{\kappa^2}{4\pi^2} \frac{\sin^2[\pi(x-x')]}{\sinh^2[\kappa(x-x')/2]} , \qquad \text{for} \quad x\, x' > 0$$

(Normal: translational invariant)

Summing up… (part 1)

- Luttinger Liquid predicts oscillatory term in correlator
- Possible to detect them in a BEC in Tonks-Girardeau regime



$$Y_2(x,x') = \frac{\kappa^2}{4\pi^2}\frac{\sin^2[\pi(x-x')]}{\sinh^2[\kappa(x-x')/2]}\,\theta(x\,x') + \frac{\kappa^2}{4\pi^2}\frac{\sin^2[\pi(x-x')]}{\cosh^2[\kappa(x+x')/2]}\,\theta(-\,x\,x')$$

Summing up… (part 2)

- RME is time-reversal invariant

  → LL in thermal equilibrium with bath due to horizon (Hartle-Hawking effect)

- BEC is time-reversal broken

  → actual Hawking radiation

- Kravtsov & Tsvelik (2001) already proposed a finite T LL for *critical non-invariant ensemble*

  → relationship between the two models?

Conclusions & Outlook

- We reproduced the asymptotic 2-point function in a Luttinger Liquid in curved space-time description
- Curved metric with horizons → Hawking radiation
- Equivalence with BEC system (oscillatory term)
- Underlying integrable model as interesting probe for emerging Quantum Gravity (transplanckian problem)
- Many unresolved questions (microscopical derivation?): nature of thermal bath

Thank you!