Truncations of random unitary matrices revisited

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joint review with H.-J. Sommers:
Non-Hermitian Random Matrix Ensembles, arXiv:0911.5645,
to appear in the Oxford Handbook of Random Matrix Theory

Setup

Choose a unitary matrix at random and partition it:

$$U = \begin{pmatrix} T & S \\ Q & R \end{pmatrix} \mapsto T \qquad T \text{ is } m \times p$$

'Choose a unitary matrix at random' - implicit referral to uniform distribution on the group surface, known as the Haar measure.

Thus, consider the unitary group $U(n)$ equipped with the normalised Haar measure $\mathrm{d}\mu_H(U)$. The property of invariance determines this measure uniquely. This can be used for sampling of the Haar distribution via Gram-Schmidt.

The Haar measure induces a probability distribution $\mathrm{d}\rho_{n,m\times p}(T)$ on truncated unitaries.

What truncations are good for?

(1) **Quantum transport problems** (Beenakker'97, poster by Nick Simm)

Additive stats of EVs of $TT^\dagger$ describe phys quantities of interest, i.e. $\operatorname{tr} TT^\dagger$ for conductance of quasi one-dimensional wires

(2) **Open chaotic sys** (Fyodorov & Sommers, '97 Życzkowski & S. '00)

Eigenvalues of T are used to model resonances

(3) **Combinatorics of vicious walkers**(Novak '09)

$< |\operatorname{tr} T|^N >_T$ enumerates configs of random-turn vicious walkers

(4) **Random determinants** (Fyodorov & K., '07), e.g.,

$$\left\langle \frac{1}{|\det(I - zA)|^{2m}} \right\rangle_A = \int \left\langle \frac{1}{\det(I - |z|^2 TT^\dagger \otimes AA^\dagger)} \right\rangle_A \mathrm{d}\rho_{n,m\times m}(T)$$

for complex random $n \times n$ matrices A with invariant distribution.

Singular values of T (1,4); eigenvalues of T (2,3)

Matrix measure

Truncation map: $U \mapsto T$, U is $n \times n$, T is $m \times p$, $m \leq p$

Have $TT^\dagger + SS^\dagger = I$ by unitarity. If $n \geq m + p$ then (generically) $SS^\dagger$ has rank m and the image of $U(n)$ is the entire matrix ball $TT^\dagger \leq I$.

Theorem 1 *(Friedman&Mello '85, Fyodorov&Sommers '03, Forrester '06)*

For $n \geq m + p$

$$\mathrm{d}\rho_{n,m\times p}(T) \propto \det(I - TT^\dagger)^{n-m-p} \chi_{TT^\dagger \leq I}(T)\mathrm{d}T$$

where dT is the Cartesian volume element in $\mathbf{C}^{m\times p}$. For invariant f

$$\int_{\mathbf{C}^{m\times p}} f(TT^\dagger)\mathrm{d}\rho_{n,m\times p}(T) = \textit{const.}\times$$

$$= \int_{\mathbf{C}^{m\times m}} f(ZZ^\dagger) \det(ZZ^\dagger)^{p-m} \det(I - ZZ^\dagger)^{n-m-p} \chi_{ZZ^\dagger \leq I}(Z) dZ$$

Matrix measure

If $n < m + p$ (e.g., deleting just a few columns/rows) then $\lambda = 1$ is an eigv of $TT^\dagger$ of multiplicity $m + p - n$. Hence the image of $U(n)$ is a set on the boundary of $TT^\dagger \leq I$.

Useful explicit expression for $\mathrm{d}\rho_{n,m\times p}(T)$ is unknown. However:

Theorem 2 *(Fyodorov & K. '07) Let $n < m + p$. Then for invariant f*

$$\int f(TT^\dagger) d\rho_{n,m\times p}(T) = \textit{const.} \times$$

$$\int f\begin{pmatrix} ZZ^\dagger & 0 \\ 0 & I \end{pmatrix} \det(ZZ^\dagger)^{p-m} \det(I - ZZ^\dagger)^{m+p-n} \chi_{ZZ^\dagger \leq I}(Z) dZ$$

(matrices T are $m \times p$ and Z are $(n-p)\times(n-p)$)

SVs of truncations: with Thms 1 and 2 in hand one can study distr. of (nontrivial) eigenvalues of $TT^\dagger$.

Joint pdf of eigenvalues of truncations of Haar unitaries

Consider square truncations T ($m \times m$), these are random contractions.

Theorem 3 *(Życzkowski & Sommers '00) The symmetrised jpdf of the EVs of T is*

$$P(z_1, \ldots, z_m) \propto \prod_{j=1}^{m} w(z_j) \prod_{1 \leq j < k \leq m} |z_j - z_k|^2$$

with weight function $w(z) = (1 - |z|^2)^{n-m-1} \chi_{|z|<1}(z)$

Ż & S didn't use $d\rho_{n,m\times m}$. Equally, (1) can be obtained via Thm 1,2, e.g. Forrester & Krishnapur '09 for $n > 2m$.

Two interesting regimes: (i) $n \to \infty$, $n - m = O(1)$ (weak non-unitarity) and (ii) $n \to \infty$, $m/(n-m) = \alpha = O(1)$ (strong non-unitarity)

Eigenvalue correlation functions

These are just marginals of the jpdf:

$$R_k(z_1,\ldots,z_k) = \frac{m!}{(m-k)!}\int \mathrm{d}^2 z_{k+1}\ldots\int \mathrm{d}^2 z_m P(z_1,\ldots,z_k,z_{k+1},\ldots,z_m)\,,$$

The EV corr fncs for truncations can be obtained by the method of OPs.

For the rotation invariant weights, $w(z)=w(|z|)$, OPs are just powers z^l: $\int d^2 z\; w(z)\, z^i z^{*j} = h_j \delta_{i,j}$, leading to

$$R_k(z_1,\ldots,z_k) = \prod_{l=1}^{m} w(z_l)\,\det(K(z_i,z_j)); \qquad K(u,v) = \sum_{l=0}^{m-1}\frac{(uv^*)^l}{h_l}$$

For truncated unitaries the sum on the rhs is the **binomial series** for $(1-uv^*)^{-(n-m+1)}$ truncated after m terms. This gives the kernel in terms of the incomplete Beta function.

Kernel

Incomplete Beta fnc:

$$I_x(a,b) = \frac{1}{B(a,b)} \int_0^x t^{a-1}(1-t)^{b-1}\, dt$$

We have

$$K(u,v) = \frac{n-m}{\pi} \frac{I_{1-uv^*}(n-m+1,m)}{(1-uv^*)^{n-m+1}}$$

This representation seems to be new. It is convenient for asymptotic analysis. Also one can handle more general $w(z) = |z|^{2p}(1-|z|^2)^q$.

Compare with the complex Ginibre ($d\mu(J) \propto e^{-\operatorname{tr} JJ^\dagger} dJ$). There $w(z) = e^{-|z|^2}$, have truncated exponential series for the kernel:

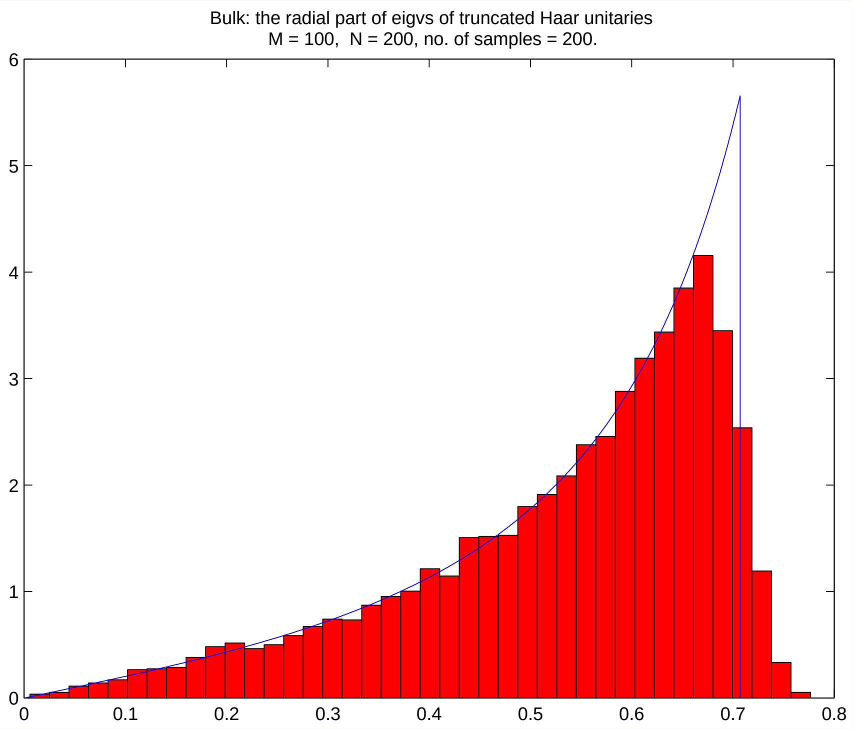
$$K(u,v) = \frac{n}{\pi} e^{uv^*} \frac{\Gamma(n,uv^*)}{\Gamma(n)} \quad \text{with} \quad \Gamma(n,x) = \int_x^\infty e^{-t} t^{n-1} dt$$

Strong non-unitarity - EV density in the bulk

Consider $n, m \to \infty$, $m/(n-m) = \alpha > 0$.

The EVs of truncated unitaries are distributed inside the disk $|z|^2 \leq \frac{\alpha}{(1+\alpha)}$ with density (Życzkowski, Sommers, '00)

$$R_1(z) \simeq \frac{m}{\pi\alpha} \frac{1}{(1-|z|^2)^2}.$$

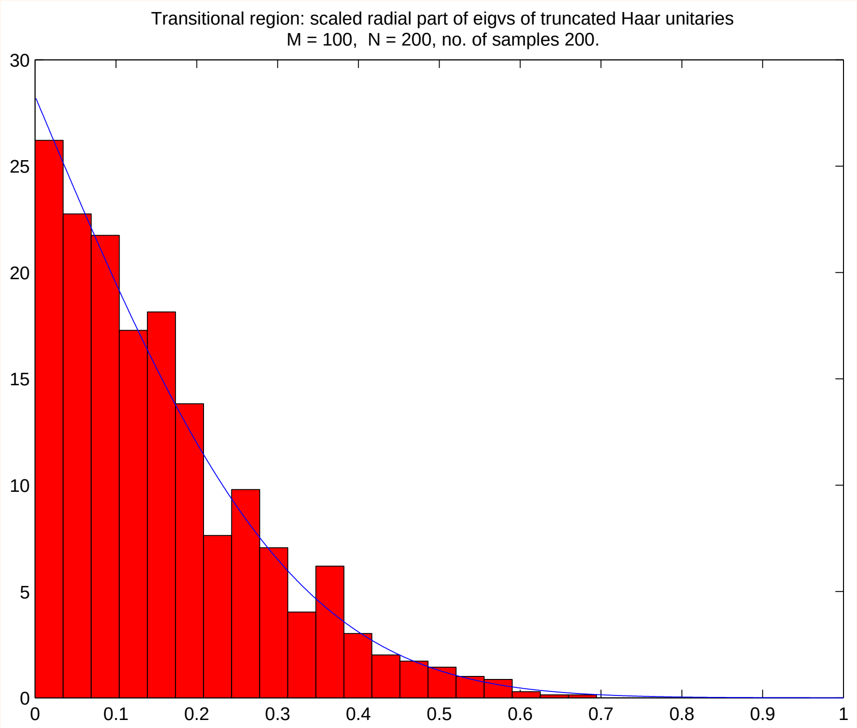


Strong non-unitarity: boundary of EV distribution

When one traverses the boundary of the support of the limiting distribution the EV density vanishes at a Gaussian rate. In the transitional region

$$R_1\left(\sqrt{\frac{\alpha}{1+\alpha}}+\frac{x}{\sqrt{m}}\right)\simeq\frac{m}{2\pi}\,\frac{(1+\alpha)^2}{\alpha}\,\operatorname{erfc}\left(\sqrt{2}\,\frac{1+\alpha}{\sqrt{\alpha}}\,x\right)$$

Same Erfc Law as in Ginibre (also poster by Navinder Singh). The average no. of EVs outside the support $\simeq\sqrt{m(1+\alpha)/(2\pi)}$.



Strong non-unitarity, locally at the origin

Scale z with the mean dist between EVs, $z_j = \frac{u_j}{\sqrt{R_1(0)}}, |u_j| = O(1)$.

Rescaled corr fncs are given by Ginibre's: $\mathrm{e}^{-\pi\sum_j |u_j|^2} \det(\mathrm{e}^{\pi u_i u_j^*})$.

Density of nearest-neighbour distances: $p(s) = -\frac{\mathrm{d}}{\mathrm{d}s} H(s)$ where

$$H(s) = \frac{m}{R_1(0)} \int \mathrm{d}^2 z_2 \cdots \int \mathrm{d}^2 z_m P(0, z_2, \ldots, z_m) \prod_{j=2}^{m} (1 - \chi_D(z_j)) \, .$$

(cond. prob. given one EV at $z = 0$ all others outside the disk D $|z| < s$)

$H(s)$ can be found by making use of rotational invariance

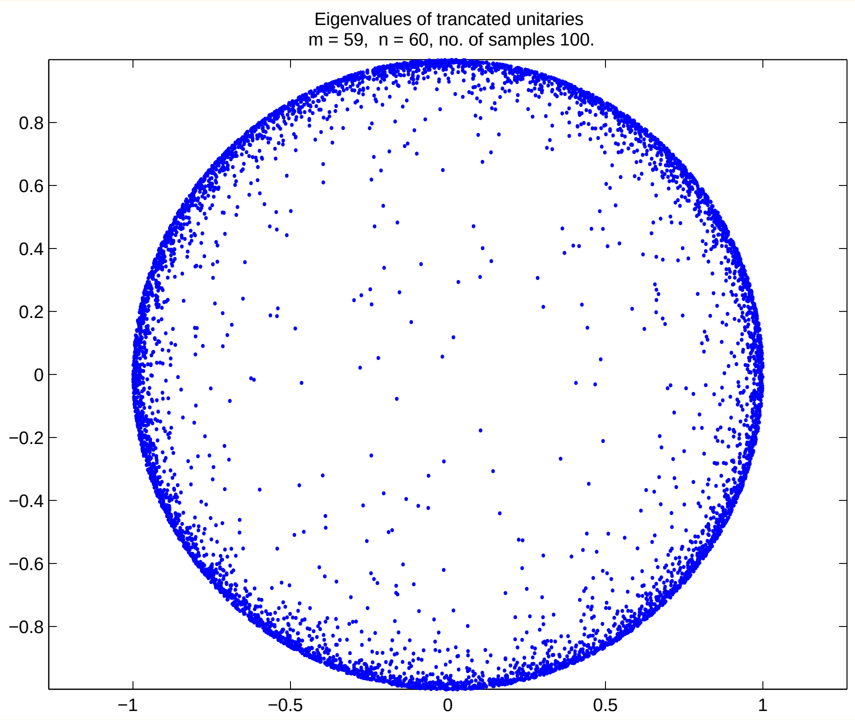
$$H(s) = \prod_{j=1}^{m-1} (I_{1-s^2}(n-m, j+1)) \simeq \prod_{j=1}^{\infty} \frac{\Gamma(j+1, \pi x^2)}{\Gamma(j+1)}, \quad s = \frac{x}{\sqrt{R_1(0)}},$$

Have **cubic law** of EV repulsion as in Ginibre (Grobe,Haake&Sommers'88)

Weak non-unitarity

Assume $n = m + l, m \to \infty$, l is finite (delete a few rows& columns)

In this limit the EVs of truncated Haar unitaries lie close to the unit circle with the typical distance being $O(1/m)$.



Weak non-unitarity: EV density and correlations

Scaling z accordingly, $z_j = \left(1 - \frac{y_j}{m}\right) \mathrm{e}^{\mathrm{i}\varphi_0 + \mathrm{i}\frac{\varphi_j}{m}}$, one finds the EV density

$$R_1(z) \simeq \frac{m^2}{\pi} \frac{(2y)^{l-1}}{(l-1)!} \int_0^1 \mathrm{e}^{-2yt} t^l \, \mathrm{d}t \,, \quad m \to \infty \text{ and } l \text{ is finite.}$$

(Życzkowski & Sommers, ’00) and correlations

$$R_k(z_1, \ldots, z_k) \simeq \left(\frac{m^2}{\pi}\right)^k \prod_{j=1}^{k} \frac{(2y_j)^{l-1}}{(l-1)!} \det\left(\int_0^1 \mathrm{e}^{-(y_i + y_j + \mathrm{i}(\varphi_i - \varphi_j))t} \, t^l \, \mathrm{d}t\right)$$

This is a particular case of a ’universal’ expression describing EV correlations for random contractions (Fyodorov & Sommers ’03).

Interestingly, a different ensemble, $J = H + \mathrm{i}\gamma W$, leads to the same form of correlations (Fyodorov & K. ’99). Here H is drawn from the GUE, $\gamma > 0$ and W is a diagonal matrix with l 1’s and m zeros.

Conclusion

A simple model leading to universal eigenvalue statistics.