Bulk universality for Wigner matrices

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Goal: describe statistical properties of eigenvalues of $N \times N$ matrices with random entries, in the limit $N \to \infty$.

Here we will restrict attention to **Wigner matrices** (entries are independent and identically distributed random variables).

Wigner matrices have been introduced by Wigner to describe excitation spectrum of heavy nuclei.

First step towards understanding of more complicated ensembles of random matrices (i.e. band matrices) and **random Schrödinger operators** (Anderson model) in the metallic phase.

Hermitian Wigner Matrices: $N \times N$ matrices $H = (h_{kj})_{1 \le k,j \le N}$ such that $H^* = H$ and

$$h_{kj} = \frac{1}{\sqrt{N}} \left(x_{kj} + i y_{kj} \right) \qquad \text{for all } 1 \le k < j \le N$$

$$h_{kk} = \frac{2}{\sqrt{N}} x_{kk} \qquad \text{for all } 1 \le k \le N$$

where x_{kj}, y_{kj} and x_{kk} $(1 \le k \le N)$ are iid with

$$\mathbb{E}\, x_{jk} = 0 \qquad \text{and} \qquad \mathbb{E}\, x_{jk}^2 = \frac{1}{2} \qquad \left(\Rightarrow \quad \mathbb{E}\, |h_{jk}|^2 = \frac{1}{N} \right)$$

Remark: similar definition for real symmetric ensembles.

Remark: scaling so that eigenvalues remain bounded as $N \to \infty$.

$$\mathbb{E} \sum_{\alpha=1}^{N} |\lambda_\alpha|^2 = \mathbb{E}\, \text{Tr}\, H^2 = \mathbb{E} \sum_{j,k=1}^{N} |h_{jk}|^2 = N^2\, \mathbb{E}\, |h_{jk}|^2$$

$$\Rightarrow \quad \mathbb{E}|h_{jk}|^2 = O(N^{-1})$$

Gaussian Unitary Ensemble (GUE): simplest example of hermitian Wigner ensemble. Probability density given by

$$P(H)\mathrm{d}H = \text{const} \cdot e^{-\frac{N}{2}\mathrm{Tr}\,(H^2)}\mathrm{d}H$$

Big advantage: joint eigenvalue distribution is explicit

$$p(\lambda_1, \ldots, \lambda_N) = \text{const} \cdot \prod_{i<j}^{N} (\lambda_i - \lambda_j)^2 \, e^{-\frac{N}{2}\sum_{j=1}^{N} \lambda_j^2} .$$

Dyson's sine-kernel distribution for GUE: using the explicit formula for density, local eigenvalue statistics can be computed in limit $N \to \infty$. Let

$$p^{(k)}(\lambda_1, \ldots, \lambda_k) = \int \mathrm{d}\lambda_{k+1} \ldots \mathrm{d}\lambda_N \, p(\lambda_1, \ldots, \lambda_N)$$

be the k-point correlation function. Then

$$\frac{1}{\varrho_{sc}^k(E)} \, p^{(k)}\Big(E + \frac{x_1}{N\varrho_{sc}(E)}, .., E + \frac{x_k}{N\varrho_{sc}(E)}\Big) \to \det\left(\frac{\sin(\pi(x_i - x_j))}{\pi(x_i - x_j)}\right)_{i,j\le k}$$

Semicircle Law (Wigner, 1955): for any $\delta > 0$,

$$\lim_{\eta \to 0} \lim_{N \to \infty} \mathbb{P} \left(\left| \frac{\mathcal{N}[E - \frac{\eta}{2}; E + \frac{\eta}{2}]}{N\eta} - \rho_{\text{sc}}(E) \right| \geq \delta \right) = 0$$

where

$$\mathcal{N}[I] = \text{number of eigenvalues in interval } I$$
$$\rho_{\text{sc}}(E) = \frac{1}{2\pi}\sqrt{1 - E^2/4}.$$

Remark 1: Wigner result concerns the macroscopic density, that is the density in intervals containing order N eigenvalues.

Remark 2: semicircle independent of distribution of entries.

Local Semicircle Law: what about density in small intervals?

Theorem 1 [Erdős-S.-Yau, 2008]: Suppose $\mathbb{E}\, e^{\nu x_{ij}^2} < \infty$ for some $\nu > 0$, and fix $|E| < 2$. Then, for any $\delta > 0$,

$$\lim_{K\to\infty}\lim_{N\to\infty}\mathbb{P}\left(\left|\frac{\mathcal{N}\left[E-\frac{K}{2N}; E+\frac{K}{2N}\right]}{K}-\rho_{sc}(E)\right|\geq\delta\right)=0$$

More precisely, we show that

$$\mathbb{P}\left(\left|\frac{\mathcal{N}\left[E-\frac{K}{2N}; E+\frac{K}{2N}\right]}{K}-\rho_{sc}(E)\right|\geq\delta\right)\leq Ce^{-c\delta\sqrt{K}}$$

for all $K > 0$, uniformly in $N > N_0(\delta)$.

On intermediate scales, if $\eta(N) \to 0$ such that $N\eta(N) \to \infty$, we have

$$\lim_{N\to\infty}\mathbb{P}\left(\left|\frac{\mathcal{N}\left[E-\frac{\eta(N)}{2}; E+\frac{\eta(N)}{2}\right]}{N\eta(N)}-\rho_{sc}(E)\right|\geq\delta\right)=0$$

Main ingredients of proof: upper bound on density and fixed point equation for Stieltjes transform.

Upper bound: observe that

$$\mathcal{N}[E-\eta/2, E+\eta/2] = \sum_\alpha \mathbf{1}(|\lambda_\alpha - E| \le \eta)$$
$$\le \sum_\alpha \frac{\eta^2}{(\lambda_\alpha - E)^2 + \eta^2} = \eta \,\mathrm{Im} \sum_\alpha \frac{1}{\lambda_\alpha - E - i\eta}$$

and hence

$$\rho = \frac{\mathcal{N}[E-\eta/2, E+\eta/2]}{N\eta}$$
$$\le \frac{1}{N} \mathrm{Im}\ \mathrm{Tr}\, \frac{1}{H-E-i\eta} = \frac{1}{N} \mathrm{Im} \sum_{j=1}^{N} \frac{1}{H-E-i\eta}(j,j)$$

We bound, for example, the $(1,1)$-element of the diagonal.

Decomposing H as

$$H = \begin{pmatrix} h_{11} & \mathbf{a}^* \\ \mathbf{a} & B \end{pmatrix}$$

we find (Feshbach map)

$$\frac{1}{H-z}(1,1) = \frac{1}{h_{11} - z - \mathbf{a} \cdot (B-z)^{-1}\mathbf{a}} = \frac{1}{h_{11} - z - \frac{1}{N}\sum_\alpha \frac{\xi_\alpha}{\lambda_\alpha - z}}$$

with

$$\xi_\alpha = N|\mathbf{a} \cdot \mathbf{u}_\alpha|^2 \qquad \Rightarrow \qquad \mathbb{E}\,\xi_\alpha = 1$$

where λ_α and $\mathbf{u}_\alpha$ are eigenvalues and eigenvectors of B.

We conclude that

$$\text{Im}\ \frac{1}{H-E-i\eta}(1,1) \le \frac{1}{\eta + \frac{1}{N}\sum_\alpha \frac{\eta}{(\lambda_\alpha - E)^2 + \eta^2}} \le \frac{N\eta}{\sum_{\alpha : |\lambda_\alpha - E| \le \eta} \xi_\alpha} \le \frac{C}{\rho}$$

Fixed point equation: we consider the Stieltjes transform

$$m_N(z) = \frac{1}{N}\text{Tr}\,\frac{1}{H-z}, \qquad m_{\text{sc}}(z) = \int \text{d}y \frac{\rho_{\text{sc}}(y)}{y-z}$$

Convergence of the density follows if we can prove that

$$m_N(z) \to m_{\text{sc}}(z), \qquad \text{for Im } z = \eta \geq K/N.$$

The Stieltjes transform m_{sc} solves the fixed point equation

$$m_{\text{sc}}(z) + \frac{1}{z + m_{\text{sc}}(z)} = 0$$

It is enough to show that, with high probability,

$$\left| m_N(z) + \frac{1}{z + m_N(z)} \right| \leq \delta$$

To this end, we use again

$$m_N(z) = \frac{1}{N} \sum_j \frac{1}{h_{jj} - z - \frac{1}{N}\sum_\alpha \frac{\xi_\alpha^{(j)}}{\lambda_\alpha^{(j)} - z}}$$

Delocalization of eigenvectors: let $\mathbf{v} = (v_1, \ldots, v_N)$ be an ℓ_2-normalized vector in $\mathbb{C}^N$. Distinguish two extreme cases:

Complete localization: one large component, for example

$$\mathbf{v} = (1, 0, \ldots, 0) \qquad \Rightarrow \qquad \|\mathbf{v}\|_p = 1, \text{ for all } 2 < p \le \infty$$

Complete delocalization: all components have same size,

$$\mathbf{v} = (N^{-1/2}, \ldots, N^{-1/2}) \quad \Rightarrow \quad \|\mathbf{v}\|_p = N^{-1/2+1/p} \ll 1$$

.

Theorem [Erdős-S.-Yau, 2008]: Suppose $\mathbb{E}\, e^{\nu x_{ij}^2} < \infty$ for some $\nu > 0$, fix $|E| < 2$, $K > 0$, $2 < p < \infty$. Then

$$\mathbb{P}\Big(\exists \mathbf{v} : H\mathbf{v} = \mu\mathbf{v}, |\mu - E| < \frac{K}{N}, \|\mathbf{v}\|_2 = 1, \|\mathbf{v}\|_p \ge M N^{-\frac{1}{2}+\frac{1}{p}}\Big) \le C e^{-c\sqrt{M}}$$

for all M, for N large enough.

Idea of proof: we write $\mathbf{v} = (v_1, \mathbf{w})$. Hence $H\mathbf{v} = \mu\mathbf{v}$ implies

$$\begin{pmatrix} h-\mu & \mathbf{a}^* \\ \mathbf{a} & B-\mu \end{pmatrix} \begin{pmatrix} v_1 \\ \mathbf{w} \end{pmatrix} \quad \Rightarrow \quad \mathbf{w} = v_1(\mu - B)^{-1}\mathbf{a}$$

By normalization

$$1 = v_1^2 + \mathbf{w}^2 \quad \Rightarrow \quad |v_1|^2 = \frac{1}{1 + \frac{1}{N}\sum_\alpha \frac{\xi_\alpha}{(\mu-\lambda_\alpha)^2}} \qquad (\xi_\alpha = N|\mathbf{a}\cdot\mathbf{u}_\alpha|^2),$$

where λ_α and $\mathbf{u}_\alpha$ are the eigenvalues and the eigenvectors of B.

$$|v_1|^2 \le \frac{1}{\frac{1}{N\eta^2} \sum_{\alpha:|\lambda_\alpha-\mu|\le\eta} \xi_\alpha}$$

Choosing $\eta = K/N$, for a sufficiently large $K > 0$, we find

$$|v_1|^2 \le \frac{K^2}{N} \frac{1}{\sum_{\alpha:|\lambda_\alpha-\mu|\le K/N} \xi_\alpha} \le c\,\frac{K}{N}$$

with high probability, because, by the local semicircle law, there must be order K eigenvalues λ_α with $|\lambda_\alpha - \mu| \le K/N$. $\square$

Level Repulsion [Erdős-S.-Yau, 2008]: Suppose $\mathbb{E}\, e^{\nu x_{ij}^2} < \infty$ for some $\nu > 0$, fix $|E| < 2$.

Fix $k \geq 1$, and assume that the probability density $h(x) = e^{-g(x)}$ of the matrix entries satisfies the bound

$$\left|\widehat{h}(p)\right| \leq \frac{1}{(1 + Cp^2)^{\sigma/2}}, \quad \left|\widehat{hg''}(p)\right| \leq \frac{1}{(1 + Cp^2)^{\sigma/2}} \quad \text{for } \sigma \geq 5 + k^2.$$

Then there exists a constant $C_k > 0$ such that

$$\mathbb{P}\left(\mathcal{N}\left[E - \frac{\varepsilon}{2N}; E + \frac{\varepsilon}{2N}\right] \geq k\right) \leq C_k\, \varepsilon^{k^2}$$

for all N large enough, and all $\varepsilon > 0$.

Remark: for GUE, we have

$$p(\lambda_1, \ldots, \lambda_N) \simeq \prod_{i<j} (\lambda_i - \lambda_j)^2 \quad \Rightarrow \quad \mathbb{P}(\mathcal{N}_\varepsilon \geq k) \simeq \varepsilon^{k^2}$$

Universality: local eigenvalue statistics in the limit $N \to \infty$ is expected to depend only on symmetry, but to be independent of probability law of matrix entries.

Remark: universality at the edges of the spectrum was established by Soshnikov in 1999 using the moment method. Here I will consider universality in the bulk of the spectrum.

In 2001, Johansson established the validity of bulk universality for ensembles of hermitian Wigner matrices with a Gaussian component.

Johansson's approach: consider matrices of the form

$$H = H_0 + t^{\frac{1}{2}} V$$

where V is a GUE-matrix, and H_0 is an arbitrary Wigner matrix.

The matrix H can be obtained by letting every entry of H_0 evolve under a Brownian motion up to time t (more prec. t/N).

The distribution of the eigenvalues of the matrix evolves then according to Dyson's Brownian motion

$$\mathrm{d}\lambda_\alpha = \frac{\mathrm{d}B_\alpha}{\sqrt{N}} + \frac{1}{N} \sum_{\beta \neq \alpha} \frac{1}{\lambda_\alpha - \lambda_\beta} \mathrm{d}t, \qquad 1 \leq \alpha \leq N$$

where $\{B_\alpha : 1 \leq \alpha \leq N\}$ is a collection of independent Brownian motion.

The joint probability distribution of the eigenvalues $\mathbf{x} = (x_1, \ldots, x_N)$ of H is

$$p(\mathbf{x}) = \int \mathrm{d}\mathbf{y}\; q_t(\mathbf{x};\mathbf{y})\; p_0(\mathbf{y})$$

where p_0 is the distribution of the eigenvalues $\mathbf{y} = (y_1, \ldots, y_N)$ of H_0 and

$$q_t(\mathbf{x};\mathbf{y}) = \frac{N^{N/2}}{(2\pi t)^{N/2}} \frac{\Delta_N(\mathbf{x})}{\Delta_N(\mathbf{y})} \; \det\left(e^{-N(x_j-y_k)^2/2t}\right)_{j,k=1}^{N},$$

with the Vandermonde determinant

$$\Delta(\mathbf{x}) = \prod_{i<j}^{N} (x_i - x_j) = \det \begin{pmatrix} 1 & 1 & \ldots & 1 \\ x_1 & x_2 & \ldots & x_N \\ \ldots & \ldots & \ldots & \ldots \\ x_1^N & x_2^N & \ldots & x_N^N \end{pmatrix}$$

This can be proven using the Harish-Chandra/Itzykson-Zuber formula

$$\int_{U(N)} e^{-\frac{N}{2t}\mathrm{Tr}\,(U^* R(\mathbf{x}) U - H_0(\mathbf{y}))^2} \mathrm{d}U = \frac{1}{\Delta(\mathbf{x})\Delta(\mathbf{y})} \; \det\left(e^{-\frac{N}{2t}(x_j-y_i)^2}\right)_{1\le i,j\le N}$$

The k-point correlation function of p is therefore given by

$$p^{(k)}(x_1, \ldots, x_k) = \int q_t^{(k)}(x_1, \ldots, x_k; \mathbf{y})\, p_0(\mathbf{y}) \mathrm{d}\mathbf{y}$$

where

$$\begin{aligned} q_t^{(k)}(x_1, \ldots, x_k; \mathbf{y}) &= \int q_t(\mathbf{x}; \mathbf{y})\, \mathrm{d}x_{k+1} \ldots \mathrm{d}x_N \\ &= \frac{(N-k)!}{N!}\ \det \Big(K_{t,N}(x_i, x_j; \mathbf{y}) \Big)_{1 \le i,j \le k} \end{aligned}$$

with

$$\begin{aligned} K_{t,N}(u, v; \mathbf{y}) &= \frac{N}{(2\pi i)^2 (v-u) t} \\ &\times \int_\gamma \mathrm{d}z \int_\Gamma \mathrm{d}w\, (e^{-N(v-u)(w-r)/t} - 1) \prod_{j=1}^{N} \frac{w - y_j}{z - y_j} \\ &\times \frac{1}{w-r} \Big(w - r + z - u - \frac{t}{N} \sum_j \frac{y_j - r}{(w-y_j)(z-y_j)} \Big) e^{N(w^2 - 2vw - z^2 + 2uz)/2t} \end{aligned}$$

where γ is the union of two horizontal lines and Γ is a vertical line in the $\mathbb{C}$-plane, and $r \in \mathbb{R}$ is arbitrary.

Convergence of k-point correlation follows from

$$\frac{1}{N\varrho(u)} K_{t,N}\left(u + \frac{x_1}{N\varrho(u)}, u + \frac{x_2}{N\varrho(u)}; \mathbf{y}\right) \to \frac{\sin \pi(x_2 - x_1)}{\pi(x_2 - x_1)} \qquad \text{for a.e. } \mathbf{y}$$

To prove convergence of $K_{t,N}$ to sine-kernel Johansson uses

$$\frac{1}{N\varrho(u)} K_{t,N}\left(u, u + \frac{\tau}{N\varrho}; \mathbf{y}\right) = N \int_\gamma \frac{\mathrm{d}z}{2\pi i} \int_\Gamma \frac{\mathrm{d}w}{2\pi i} h_N(w) g_N(z,w) e^{N(f_N(w) - f_N(z))}$$

with

$$f_N(z) = \frac{1}{2t}(z^2 - 2uz) + \frac{1}{N}\sum_j \log(z - y_j)$$

$$g_N(z,w) = \frac{1}{t(w-r)}[w - r + z - u] - \frac{1}{N(w-r)} \sum_j \frac{y_j - r}{(w - y_j)(z - y_j)}$$

$$h_N(w) = \frac{1}{\tau}\left(e^{-\tau(w-r)/t\varrho} - 1\right)$$

and performs a detailed asymptotic saddle analysis.

Beyond Johansson: what happens if $t = t(N) \to 0$? Consider

$$t = N^{-1+\varepsilon}$$

Similar integral representation but asymptotic analysis is more delicate and requires microscopic convergence to the semicircle.

Recall that

$$\frac{1}{N\varrho(u)} K_{t,N}\Big(u, u + \frac{\tau}{N\varrho(u)}; \mathbf{y}\Big) = N \int_\gamma \frac{\mathrm{d}z}{2\pi i} \int_\Gamma \frac{\mathrm{d}w}{2\pi i} h_N(w) g_N(z,w) e^{N(f_N(w) - f_N(z))}$$

with

$$f_N(z) = \frac{1}{2t}(z^2 - 2uz) + \frac{1}{N} \sum_j \log(z - y_j)$$

Saddles are determined by the equation

$$f_N'(z) = \frac{1}{t}(z-u) + \frac{1}{N}\sum_j \frac{1}{z-y_j} = 0$$

There are two complex conjugated solutions $z = q_N^{\pm}$.

By the convergence to the semicircle on scales of order $N^{-1+\varepsilon}$, we have, with high probability,

$$q_N^{\pm} = q^{\pm} + O(tN^{-\varepsilon/2})$$

where

$$q^{\pm} = u(1-2t) \pm 2ti\sqrt{1-u^2} + O(tN^{-\varepsilon/2})$$

are the two solutions of

$$\frac{1}{t}(q^{\pm}-u) + \int \frac{\varrho_{sc}(y)}{q^{\pm}-y}\mathrm{d}y = 0$$

The integration paths can be shifted to pass through the saddles.

Only important contribution arises from z, w both close to $q_{N,\pm}$.

Contribution from saddles can be computed through local change of variable which makes the exponent quadratic (Laplace method).

As $N \to \infty$, saddle contribution leads to sine-kernel.

Theorem [Erdős-Péché-Ramirez-S.-Yau]: Let $p_N^{(k)}$ be the k-point eigenvalue correlation function for the ensemble $H = H_0 + t^{1/2}V$, where H_0 is an arbitrary Wigner matrix, V is an independent GUE matrix, and $t \geq N^{-1+\varepsilon}$. Then

$$\lim_{N\to\infty} \frac{1}{\rho_{sc}^k(E)} p_N^{(k)} \left(E + \frac{x_1}{N\rho_{sc}(E)}, \ldots, E + \frac{x_k}{N\rho_{sc}(E)} \right) = \det \left(\frac{\sin(\pi(x_i - x_j))}{(\pi(x_i - x_j))} \right)_{i,j=1}^{k}$$

Time reversal to remove Gaussian part: let $h(x)$ be the density of the matrix elements of H_0.

The matrix elements of $H = H_0 + t^{\frac{1}{2}} V$ have density

$$h_t(x) = (e^{tL}h)(x), \qquad \text{with} \qquad L = \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}$$

Then

$$\int \frac{|h_t(x) - h(x)|^2}{h(x)} dx \leq Ct^2$$

Letting $F = h^{\otimes N^2}$ and $F_t = (e^{tL}h)^{\otimes N^2}$ we find

$$\int \frac{|F_t - F|^2}{F} dx_1 \dots dx_{N^2} \leq CN^2 t^2$$

It is only small for $t \ll N^{-1}$.

Hence $t = N^{-1+\varepsilon}$ is still not enough.

We would like to write

$$h = e^{tL} f \qquad \text{with} \qquad f = e^{-tL} h$$

But the heat equation cannot be reversed.

$\Rightarrow$ approximate inversion of heat semigroup

Define $v_t = (1 - tL)h$. Then

$$h_t = e^{tL} v_t = e^{tL}(1 - tL)h \simeq h + t^2 L^2 h \qquad \Big(\text{while} \quad e^{tL} h \simeq h + tLh\Big)$$

Therefore

$$\int \frac{|h_t - h|^2}{h} dx \leq C t^4$$

Hence, if $F = h^{\otimes N^2}$ and $F_t = h_t^{\otimes N^2}$, we find

$$\int \frac{|F_t - F|^2}{F} dx_1 \dots dx_{N^2} \leq C N^2 t^4 \ll 1 \qquad \text{for } t = N^{-1+\varepsilon}$$

Theorem [Erdős-Péché-Ramirez-S.-Yau]: Suppose H is a hermitian Wigner matrix, whose entries have law $g = e^{-h}$, for $h \in C^6(\mathbb{R})$. Then,

$$\lim_{N\to\infty} \frac{1}{\rho_{\rm sc}^2(E)} p_N^{(2)}\left(E + \frac{x_1}{N\rho_{\rm sc}(E)}, E + \frac{x_2}{N\rho_{\rm sc}(E)}\right) = \frac{\sin(\pi(x_1 - x_2))}{(\pi(x_1 - x_2))}$$

Shortly after we posted our result, Tao-Vu submitted a paper with same results. Combining two approaches, one can remove all conditions, at least after averaging over the variable u.

Theorem [Erdős-Ramirez-S.-Tao-Vu-Yau, 2009]: Fix $\varepsilon > 0$ and $|u_0| < 2 - \varepsilon$. Fix $k \geq 1$, then

$$\lim_{N\to\infty} \frac{1}{2\varepsilon} \int_{u_0-\varepsilon}^{u_0+\varepsilon} du \, \frac{1}{[\rho_{sc}(u)]^k} p^{(k)}\left(u + \frac{x_1}{N\rho_{sc}(u)}, \dots, u + \frac{x_k}{N\rho_{sc}(u)}\right)$$

$$= \det\left(\frac{\sin(\pi(x_i - x_j))}{\pi(x_i - x_j)}\right)_{i,j=1}^k$$

The local relaxation flow: Dyson Brownian Motion (DBM) describes evolution of eigenvalues. The equilibrium measure is the GUE measure

$$\mu(\mathbf{x})\mathrm{d}\mathbf{x} = \frac{e^{-\mathcal{H}(\mathbf{x})}}{Z}\mathrm{d}\mathbf{x}, \qquad \mathcal{H}(\mathbf{x}) = N\left[\sum_{j=1}^{N}\frac{x_j^2}{2} - \frac{2}{N}\sum_{i<j}\log|x_j - x_i|\right]$$

The evolution of an initial probability density function $f\mu$ w.r.t DBM is described by the heat equation

$$\partial_t f_t = L f_t,$$

with the generator

$$L = \sum_{i=1}^{N}\frac{1}{2N}\partial_i^2 + 2\sum_{i=1}^{N}\left(-\frac{1}{4}x_i + \frac{1}{2N}\sum_{j\neq i}\frac{1}{x_i - x_j}\right)\partial_i$$

Relaxation time of Dyson's Brownian motion given by

$$\frac{1}{2N}\nabla^2\mathcal{H} \geq O(1) \quad \Rightarrow \quad \text{relaxation on times } O(1)$$

Idea: introduce new flow with shorter relaxation time. Define

$$\widetilde{\mathcal{H}}(\mathbf{x}) = N\left[\sum_{j=1}^{N}\left(\frac{x_j^2}{2} + \frac{1}{2R^2}(x_j-\gamma_j)^2\right) - \frac{2}{N}\sum_{i<j}\log|x_j - x_i|\right]$$
$$= \mathcal{H}(\mathbf{x}) + \frac{N}{2R^2}\sum_{j=1}^{N}(x_j-\gamma_j)^2$$

where γ_j is position of the j-th eigenvalue w.r.t. semicircle law, and $R = N^{-\varepsilon} \ll 1$.

Introduce new equilibrium measure $\omega(\mathbf{x}) = e^{-\widetilde{H}(\mathbf{x})}/\widetilde{Z}$ and new evolution

$$\partial_t g_t = \widetilde{L} g_t \qquad \text{with} \qquad \widetilde{L} = L - \frac{1}{R^2}\sum_{j=1}^{N}(x_j - \gamma_j)\,.$$

Observe that

$$\frac{\nabla^2 \widetilde{\mathcal{H}}(\mathbf{x})}{N} \geq CR^{-2} \geq N^{2\varepsilon} \gg 1 \quad \Rightarrow \quad \text{relaxation on short times}$$

Hence, if $\mathcal{G}_{i,n}(\mathbf{x}) = G\Big(N(x_i - x_{i+1}), \ldots, N(x_{i+n-1} - x_{i+n})\Big)$, we find

$$\Big|\int \frac{1}{N}\sum_{i\in J} \mathcal{G}_{i,n}\mathrm{d}\omega - \int \frac{1}{N}\sum_{i\in J} \mathcal{G}_{i,n}\, g\, \mathrm{d}\omega\Big| \le C_n\Big(\frac{D_\omega(\sqrt{g})R^2}{N}\Big)^{1/2}$$

with the Dirichlet form

$$D_\omega(h) = \frac{1}{N}\sum_{j=1}^{N}\int \Big|\partial_{x_j} h\Big|^2 \,\mathrm{d}\omega$$

On other hand, if difference between generators is small, we expect $f_t\mu \simeq \omega = \psi\mu$. In fact, for $t \gg R^2$, we find that

$$D_\omega(\sqrt{f_t/\psi}) \le CN\Lambda \qquad \text{where} \quad \Lambda = \mathbb{E}_t \sum_j |x_j - \gamma_j|^2 .$$

From microscopic semicircle law, we find $\Lambda \le N^{-\varepsilon}$.

This implies universality for ensembles of the form $H_0 + t^{1/2}V$, if $t \geq N^{-\varepsilon}$, for arbitrary symmetry.

Time-reversal argument implies universality for all matrices whose entries have enough regularity.

Combining with the result of Tao-Vu, we find universality for arbitrary ensembles.

Theorem [Erdős, S., Yau, 2009]: For arbitrary hermitian, symmetric or symplectic Wigner matrices with subexponentially fast decaying entries, the weak limit as $N \to \infty$ of the averaged k-particle correlation function

$$\frac{1}{2\varepsilon}\int_{E-\varepsilon}^{E+\varepsilon} \mathrm{d}u \, \frac{1}{\rho_{\text{sc}}^k(u)} \, p_N^{(k)}\left(u + \frac{x_1}{\rho_{\text{sc}}(u)N}, \ldots, u + \frac{x_k}{\rho_{\text{sc}}(u)N}\right)$$

coincides with that of the corresponding Gaussian ensemble (GUE, GOE, or GSE) for all $|E| < 2$ and $\varepsilon > 0$.

Tao-Vu approach: let H and H' be two Wigner matrices whose entries have distribution x, y; assume that typical distance between eigenvalues is order one ($x, y \simeq \sqrt{N}$).

Assume that

$$\mathbb{E}\, x^m = \mathbb{E}\, y^m \qquad \text{for} \qquad 1 \leq m \leq 4$$

Fix $k \geq 1$ and consider a nice function $G : \mathbb{R}^k \to \mathbb{R}$. Then

$$|\mathbb{E}\, G(\lambda_{\alpha_1}(H), \ldots, \lambda_{\alpha_k}(H)) - \mathbb{E}G(\lambda_{\alpha_1}(H), \ldots, \lambda_{\alpha_k}(H))| \to 0$$

as $N \to \infty$.

Idea of proof: change one entry at the time.

$H(z) =$ matrix obtained from H replacing (i,j)-entry with z

$F(z) = G(\lambda_\alpha(H(z)))$ (we take $k = 1$)

$$F(x) = F(0) + xF'(0) + \cdots + \frac{x^5}{5!}F^{(v)}(0) + ..$$

$$F(y) = F(0) + yF'(0) + \cdots + \frac{y^5}{5!}F^{(v)}(0) + ..$$

Therefore

$$|\mathbb{E}F(x) - \mathbb{E}F(y)| \leq \mathbb{E}\,|x|^5 F^{(v)}(0)$$

Observe

$$\mathbb{E}|x|^5 \simeq N^{5/2} \qquad \text{but} \qquad F^{(m)}(0) \simeq N^{-m}$$

In fact

$$F'(0) = G'(\lambda_\alpha(H)) \cdot \frac{\partial \lambda_\alpha}{\partial h_{ij}} = G'(\lambda_\alpha(H)) \cdot \mathbf{v}_\alpha(i)\mathbf{v}_\alpha(j) \simeq N^{-1}$$

Hence

$$|\mathbb{E}F(x) - \mathbb{E}F(y)| \leq CN^{-5/2}$$

Repeating this argument N^2 times, we can replace all entries of H; the total error is $O(N^{-1/2})$.

Universality (Tao-Vu): for given H, find Johansson matrix

$$H_t = e^{-t/2} H_0 + (1 - e^{-t})^{1/2} V$$

such that H and H_t have four matching moments.

This is only possible if entries are supported on at least 3 points.

Universality (Erdős-Ramirez-S.-Tao-Vu-Yau): compare H with the evolved matrix

$$H_t = e^{-t/2} H + (1 - e^{-t})^{1/2} V$$

with $t = N^{-1+\delta}$.

Moments do not match, but they are very close.