Fractal Weyl laws - beyond ballistic chaotic decay

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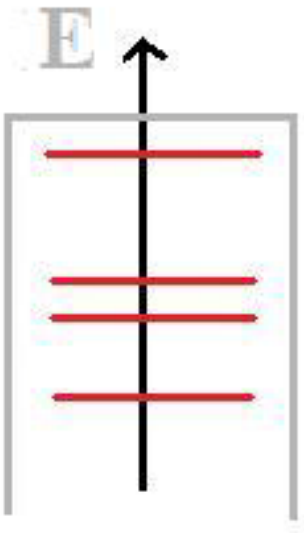
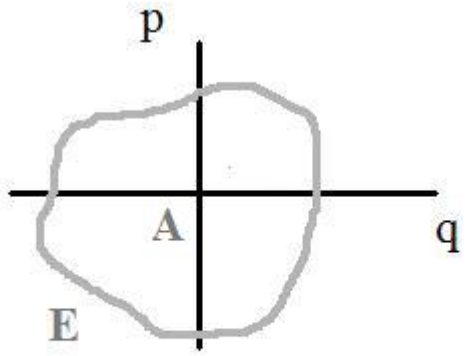
Weyl law

Qm energy levels



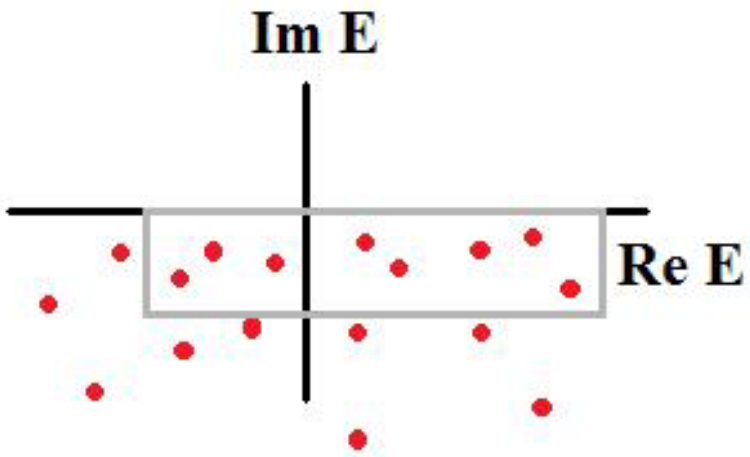
cl energy shell



Weyl: $\#(E_n<E) = A/h^f$

Open system

Complex energies E: Im $E_n=-\Gamma_n/2$

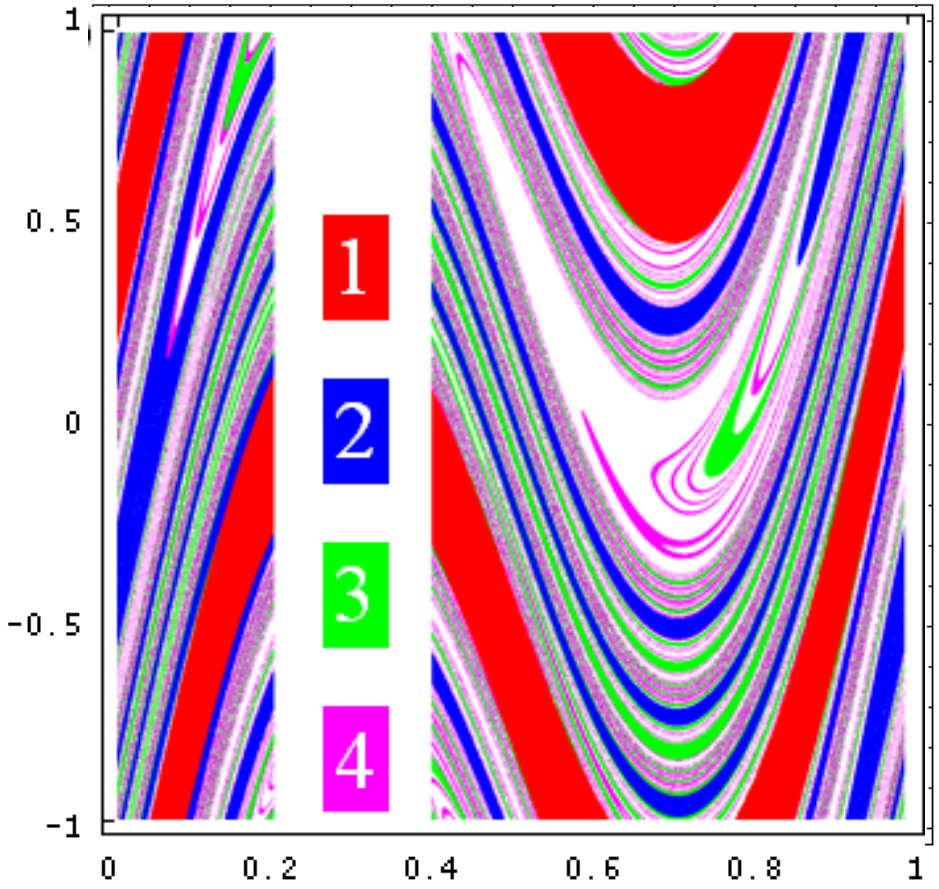


Question: #(E_n: Re $E_n<E$, $\Gamma_n<\Gamma$)

Exploit qm-cl correspondence?

round-trip operator *F, dim F=M= 1/h; opening operator P=(MxN)* ,

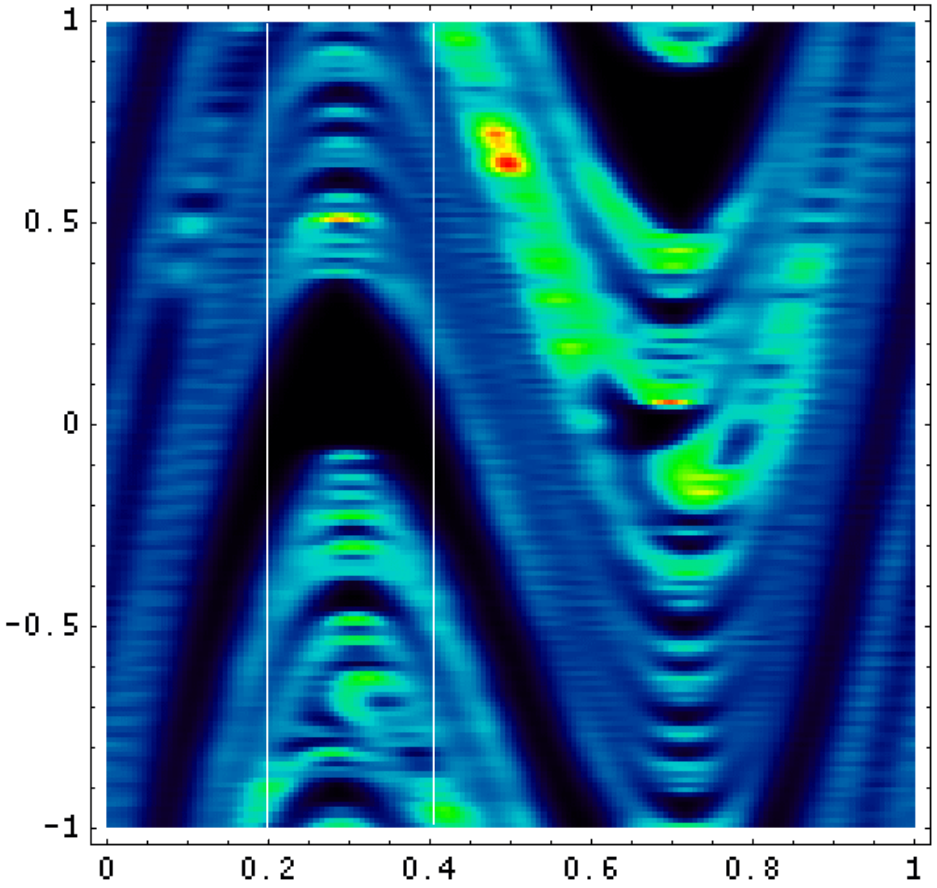
Inject a particle



t → E: Stroboscopic scattering theory

round-trip operator *F, dim F=M= 1/h; opening operator P=(MxN) ,*

internal space: projector Q=1-PP^T



inject a particle:

exit: P^TFP

$P^TF(QF)P$

$P^TF(QF)^2P$

$P^TF(QF)^3P$

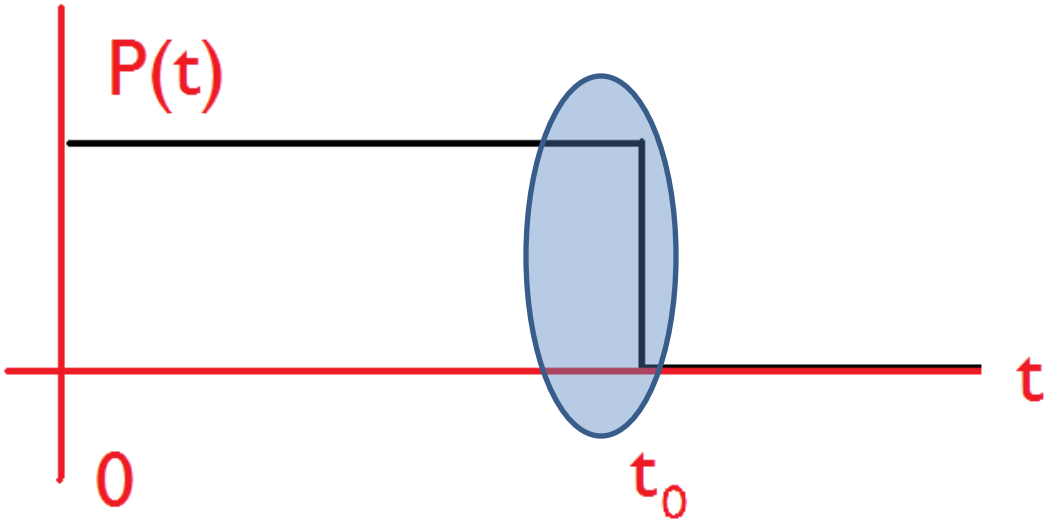
$P^TF(QF)^4P$…

$$\mathrm{FT} \Rightarrow \mathrm{S\ matrix}$$

$$S(\varepsilon) = P^T\left(e^{-i\varepsilon} - FQ\right)^{-1} FP$$

Resonances: $QFQ\psi = e^{-i\varepsilon}\psi; \quad e^{-i\varepsilon} \equiv \lambda; \quad \varepsilon = E - i\Gamma/2$

Challenge: quasi-deterministic decay



$$\lim_{\Gamma\to\infty} \frac{1}{1+e^{\Gamma(t-t_0)}}$$

- Nominally diverging decay rates: $|\lambda| = \exp(\mathrm{Im}\,\varepsilon) = 0$
- Resonance wave functions quasi-degenerate (defective eigensystem)

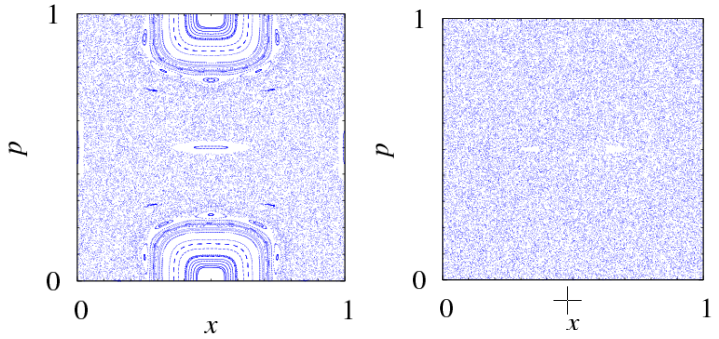
illustration: standard map/kicked rotator

(classical)

$$x_{n+1} = x_n + p_n \pmod 1$$

$$p_{n+1} = p_n + \frac{K}{2\pi}\sin(2\pi x_{n+1}) \pmod 1$$

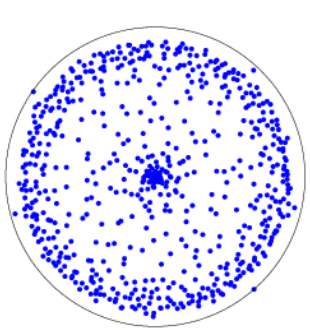
K=2 K=7.5

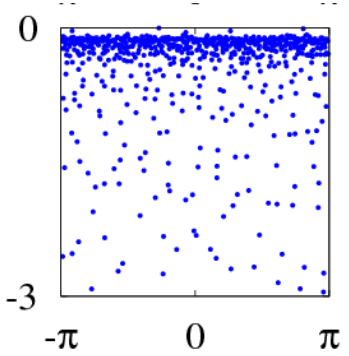


(qm)

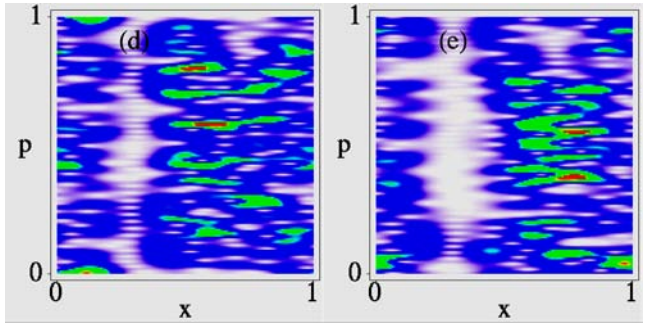
$$F_{nm} = \frac{1}{\sqrt{iM}}\exp\left[\frac{i\pi}{M}(m-n)^2 - \frac{iMK}{2\pi}\left(\cos 2\pi\frac{m}{M}\right)\right]$$

K=7.5, M=1280, N=256

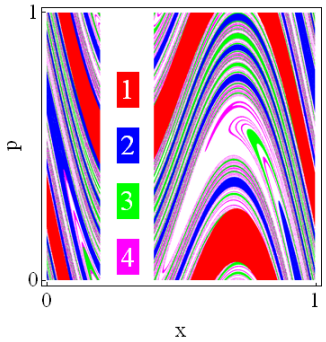




λ ε



Resonances wave functions

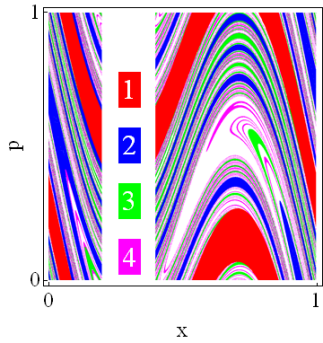


Escape zones

- Classically chaotic systems (with J Tworzydło):

  fractal Weyl law

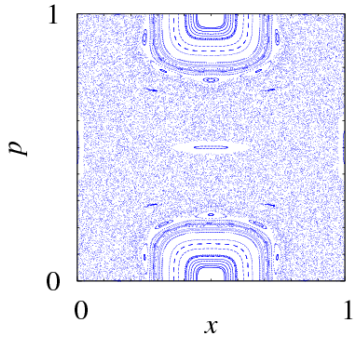
  – Goal: reinstate phase space rules



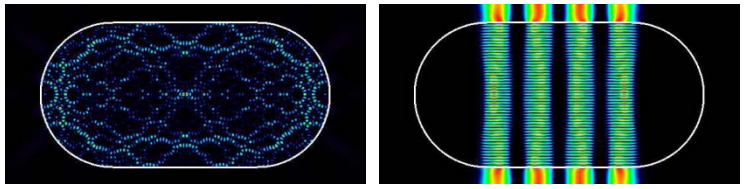
- Mixed phase space (with M Kopp):

  … fractal Weyl law …

  – Goal: test character of chaotic component

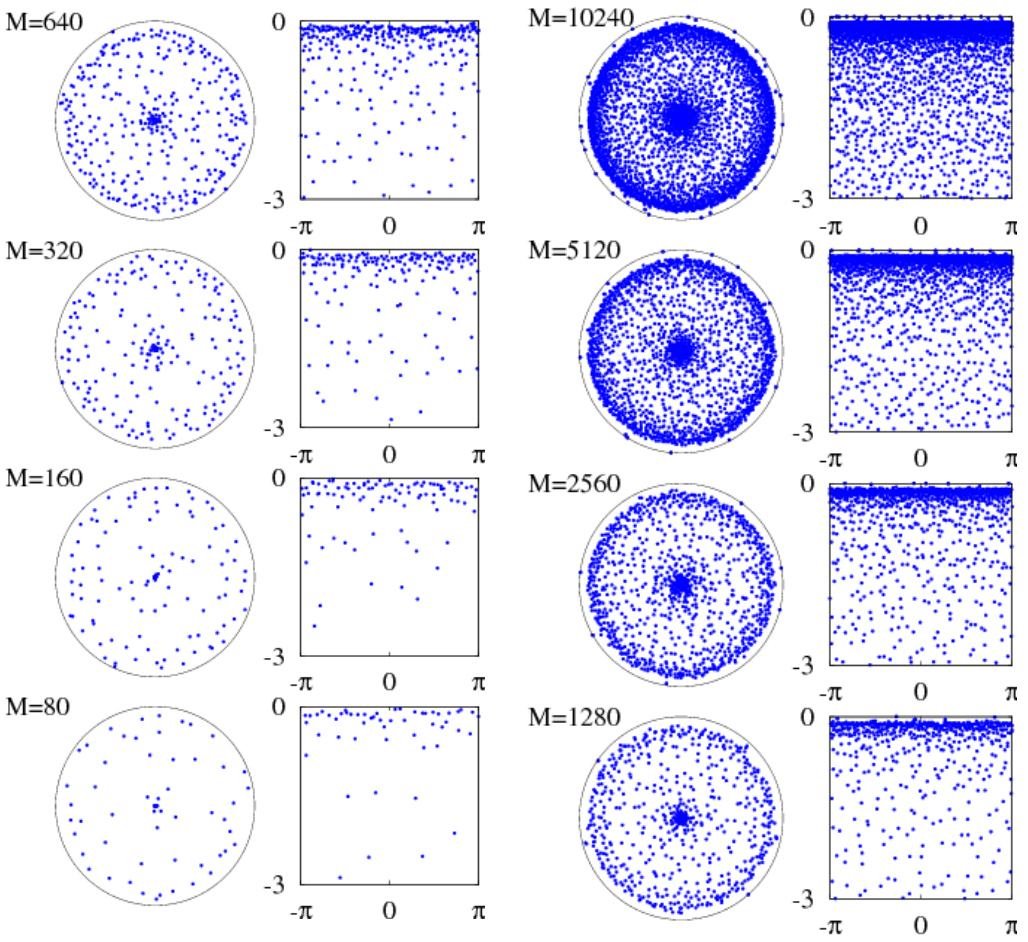


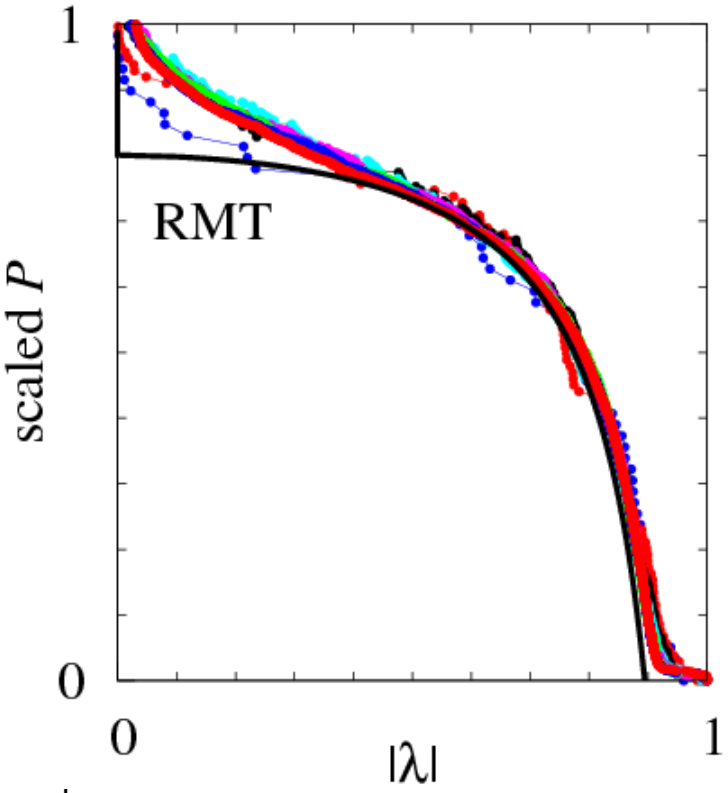
- Refractive escape (with M Kopp; J Wiersig; J Keating & M Novaes):
  – quantum dots with tunneling
  – dielectric resonators with refractive escape

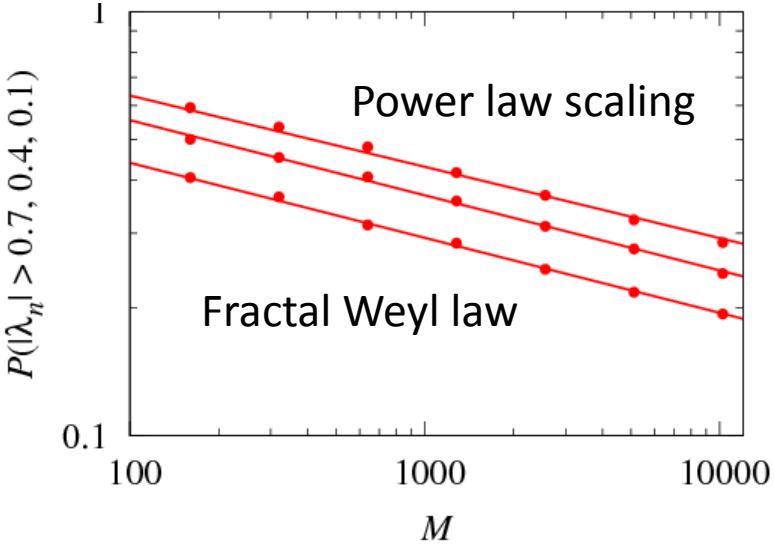


Classically chaotic systems

Resonance distribution



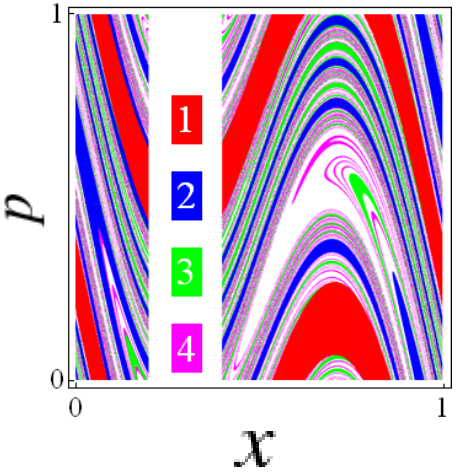




Try to count short-living states

A. identify short-lived deterministic dynamics in phase space

$$QFQ\psi_n = 0 \quad (\lambda_n = 0, \quad \Gamma_n = \infty)$$

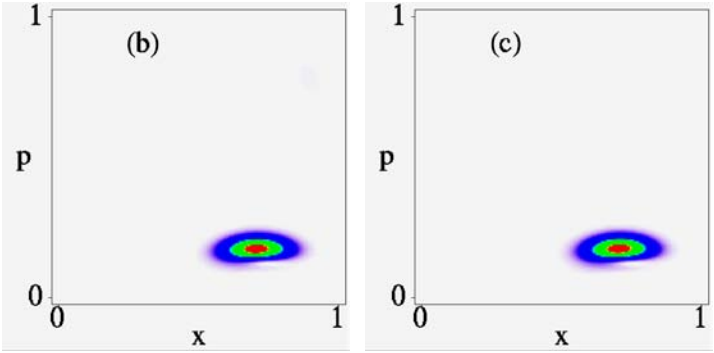


- Define $\mathcal{P}$=P^T P=1-Q
- trivially: $Q\mathcal{P}$=0 → N states on opening ($\mathcal{P}_0 = \mathcal{P}$)

- semicl.: preimage: projector $\mathcal{P}_1 = P_1 P_1^{\,T}$
- naïve Weyl: dim = area/Planck= M • area

problem: underestimates no. of states

reason: operator not self-adjoint,
states nonorthog., highly degenerate

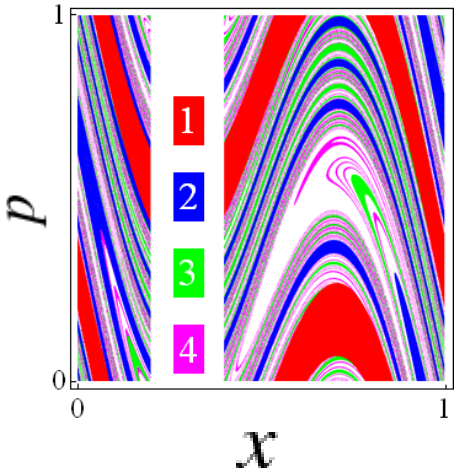


B. Cure degeneracy

$QFQ\psi_n^{(1)} = 0 \quad (\lambda_n = 0)$: consider $QFQ\psi_n^{(t+1)} = \lambda_n \psi_n^{(t+1)} + \psi_n^{(t)} = \psi_n^{(t)}$

- 2nd preimage, projector $\mathcal{P}_2 = P_2 P_2^{\mathrm{T}}$
- 3rd preimage, projector $\mathcal{P}_3 = P_3 P_3^{\mathrm{T}}$
- t^{th} preimage, projector $\mathcal{P}_t = P_t P_t^{\mathrm{T}}$
- semiclassical propagation:

$$(QFQ)^t \mathcal{P}_t = 0, \quad \mathcal{P}_t \mathcal{P}_s = 0 \quad (t \neq s)$$



C. Requires: areas $A \approx \exp(-\Lambda t) > 1/M \Rightarrow t < \frac{1}{\Lambda}\ln(M) \equiv t_{Ehr}$

Weyl: $\sum_{t < t_{Ehr}} \operatorname{rank} \mathcal{P}_t = M(1 - e^{-t_{Ehr}/t_{dwell}})$

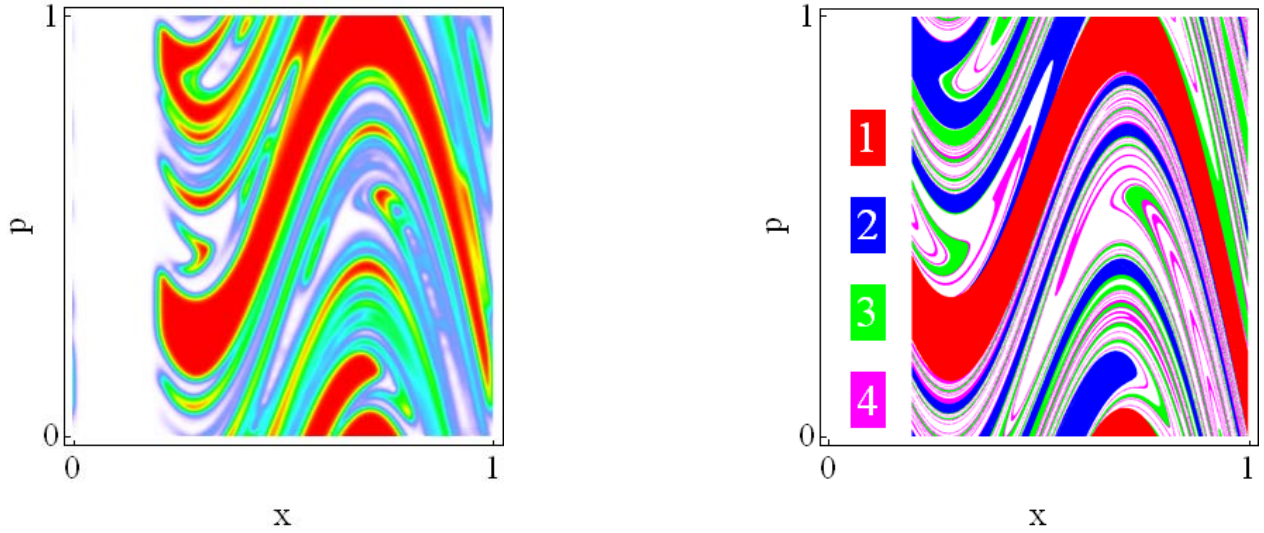
D. Remaining states (long living): $M e^{-t_{Ehr}/t_{dwell}} \propto M^{1-1/\Lambda t_{dwell}}$

What have we done? A semiclassical partial Schur decomposition!

$\mathcal{P}_t$: part of orthogonal basis U in $QFQ=UTU^+$

where T is triangular with evals on diagonal.

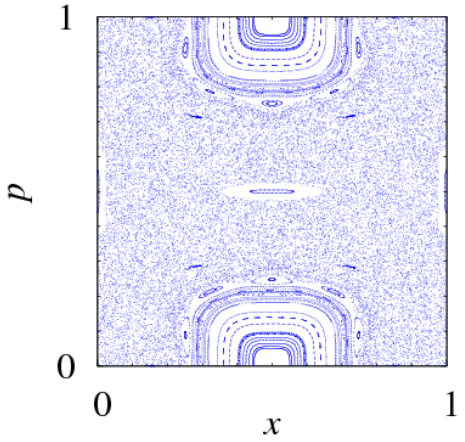
Test: Husimi rep. of Schur vectors ($|\lambda_n|<0.1$, M=1280)



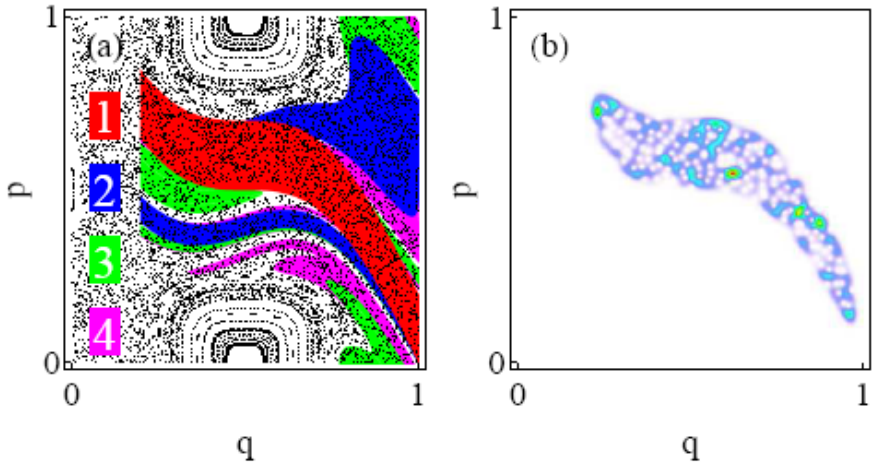
Husimi-Schur representation:

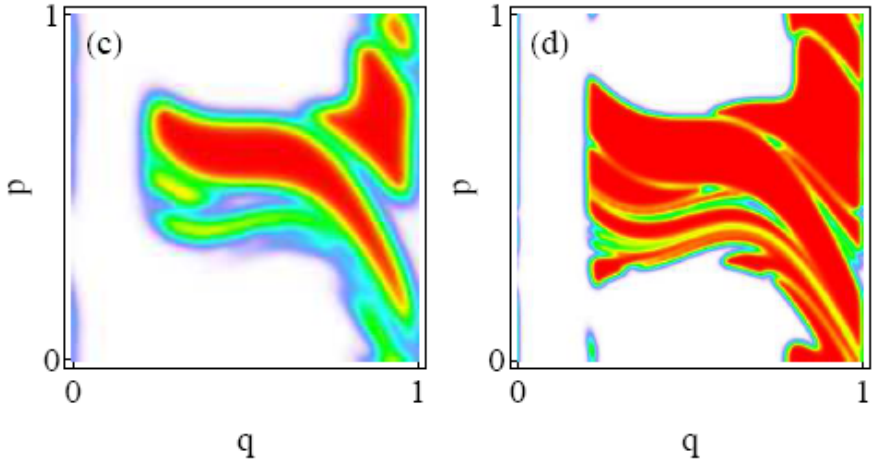
a new phase space representation for resonance eigenfunctions

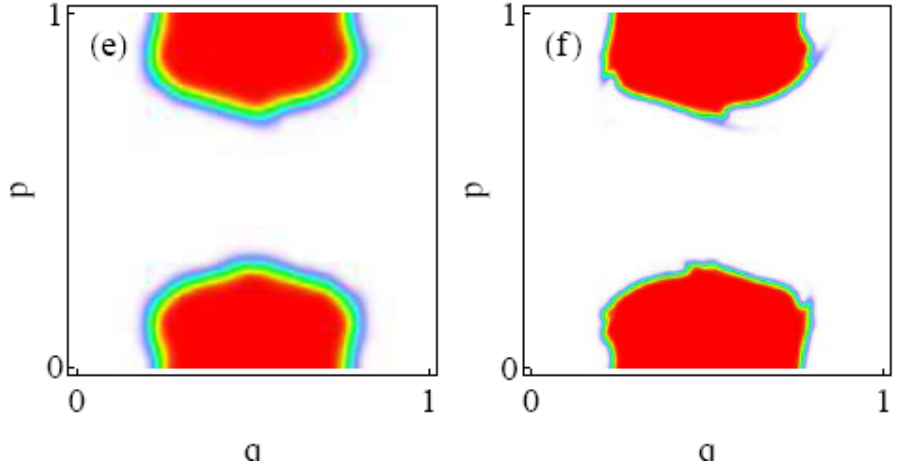
Mixed phase space



k=2.0



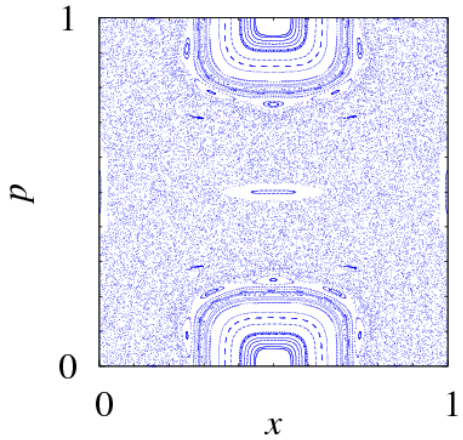


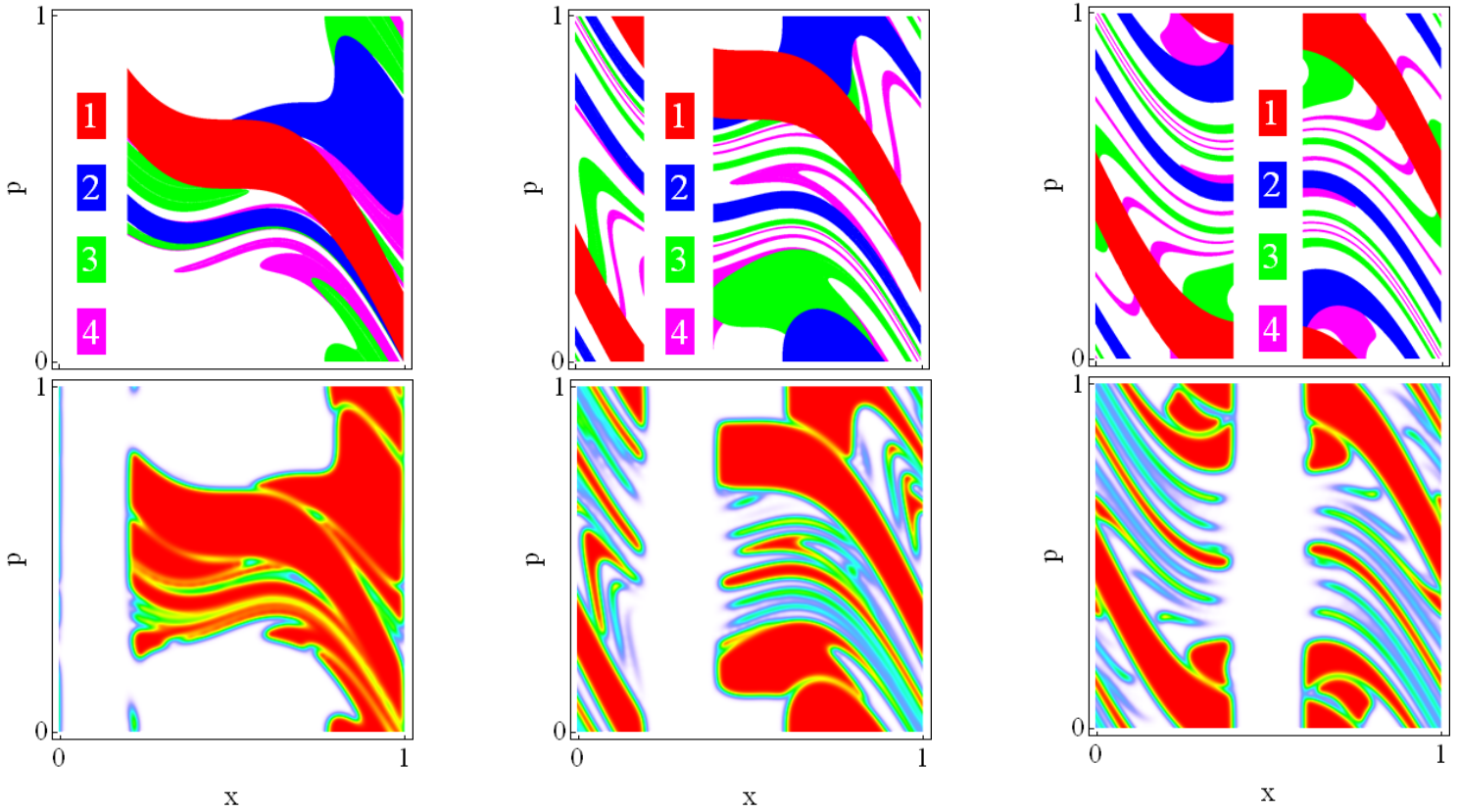


‘fast’ Husimi-Schur representation

‘slow’ Husimi-Schur representation

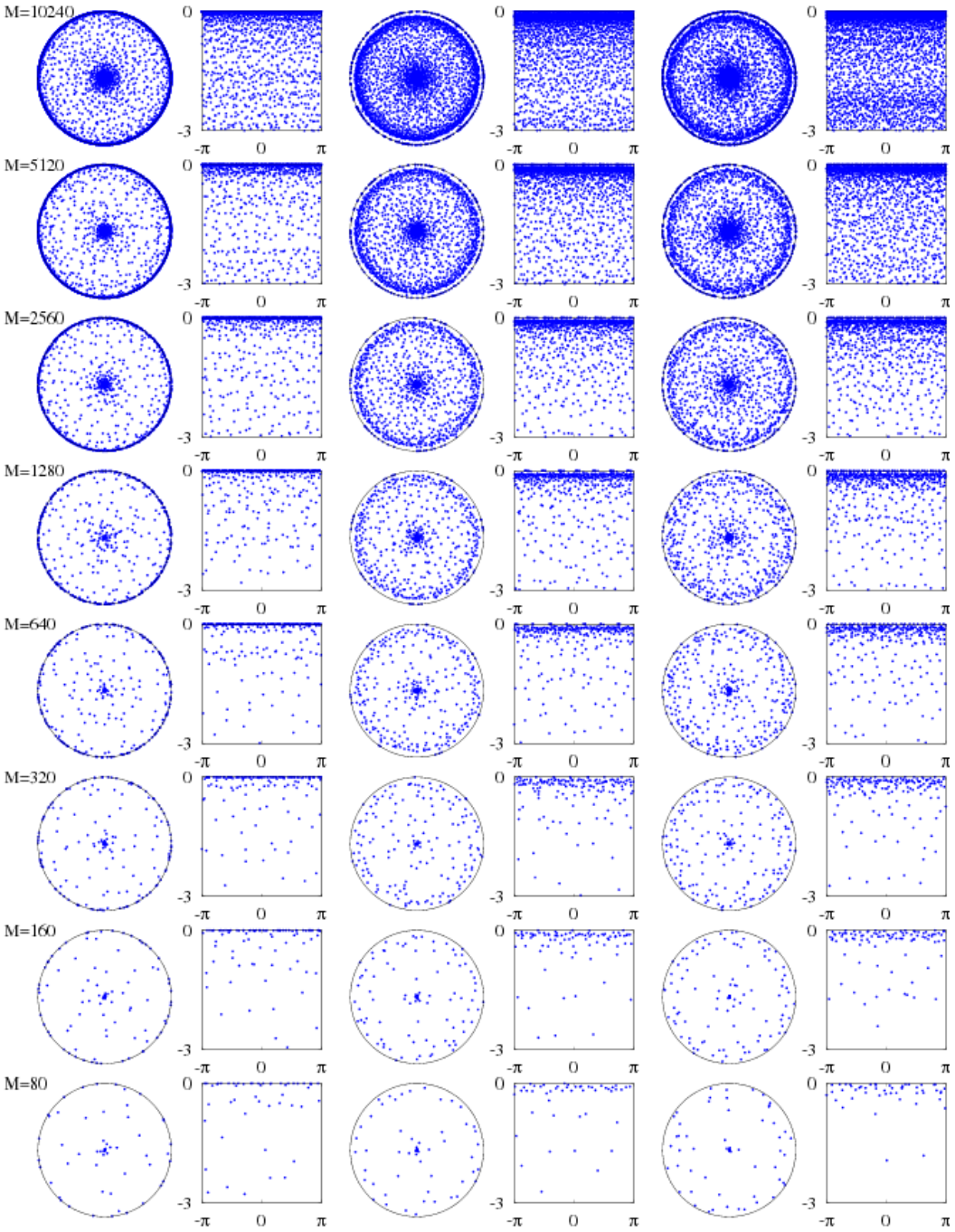
Mixed phase space





Position of leads is important; coupled islands: fast decay

Uncoupled islands: slow tunneling escape

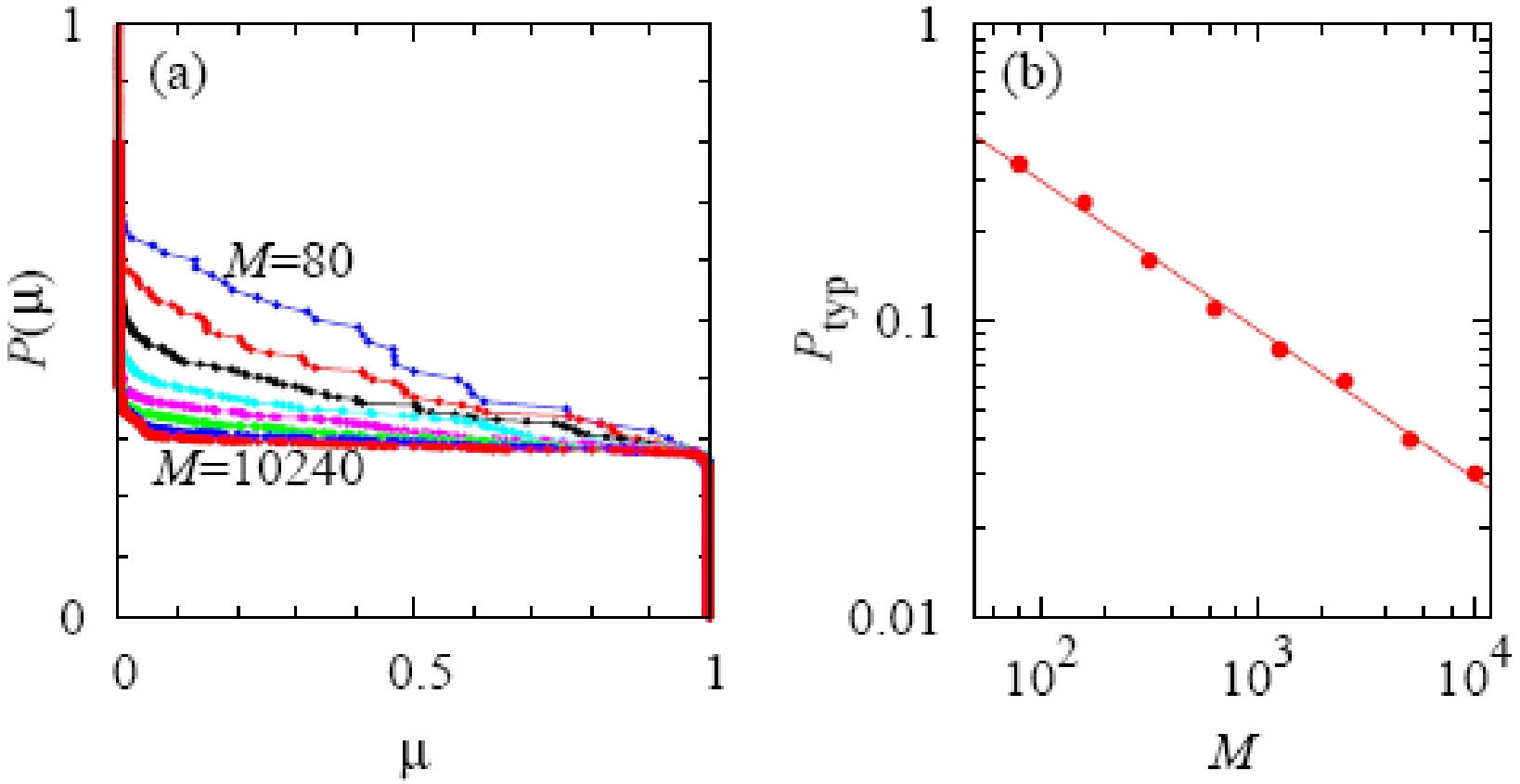


Two accumulation regions:

$$|\lambda| \approx 0{,}1$$

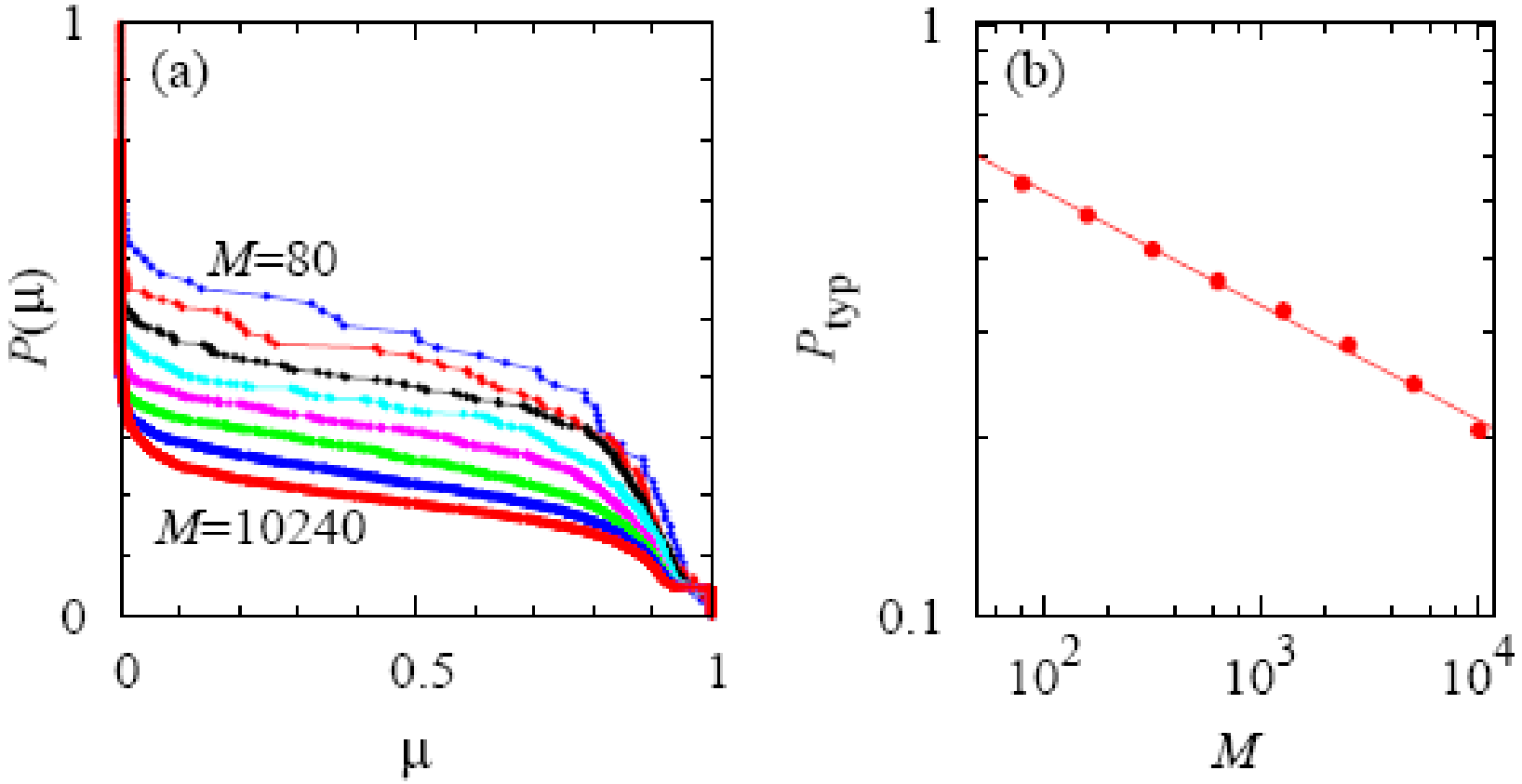
- uncoupled islands (long-living states): just the ordinary Weyl law…
- idea: fix both upper and lower cut-off of lifetimes

typical lifetimes $(0.1 < |\mu_n| < 0.98)$



opening at $0 < q < 0.2$

typical lifetimes ($0.1 < |\mu_n| < 0.98$)



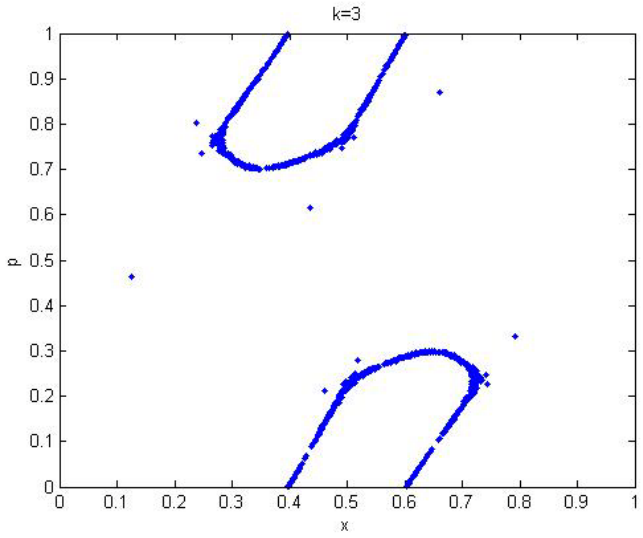
opening at $0.2 < q < 0.4$.

Slightly unexpected…

Time domain studies: classical part of mixed phase space is quite unlike a fully chaotic phase space:

Power law decay $\propto t^{-\alpha}$ instead of exponential decay $\propto \exp(-t/t_{dwell})$

Origin: sticking to islands
(see eg Cristadoro/Ketzmerick PRL 08)



Possible explanation

Areas also shrink algebraically:

algebraic loss of qm-classical correspondence $\propto t^{-\beta}$

power-law Ehrenfest time $\tau_E \propto M^{1/\beta}$

Weyl law from combination of two power laws: $\tau_E^{-\alpha} \propto M^{-\alpha/\beta}$

(in contrast to $\exp(-\tau_E/\tau_{\rm dwell}) = M^{-1/\lambda\tau_{\rm dwell}}$)

nonballistic escape I: tunnel barriers

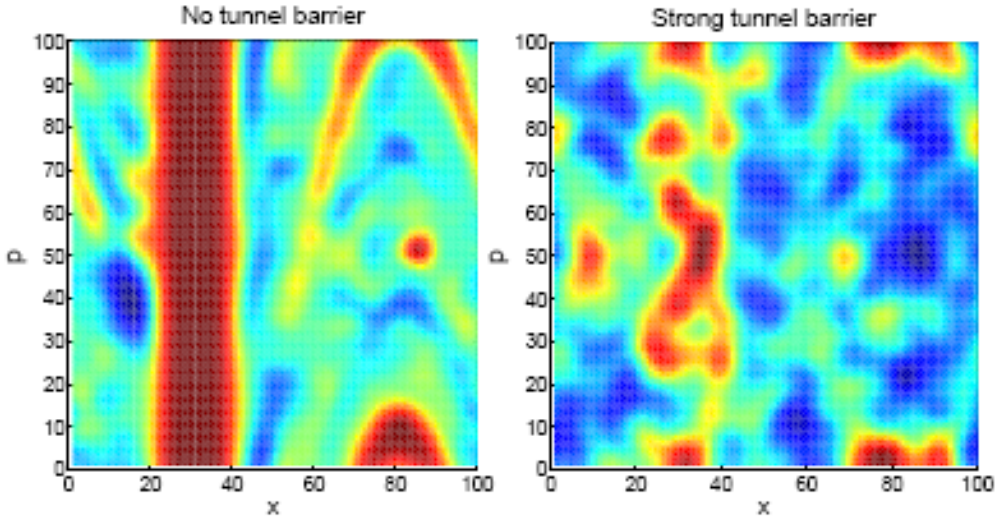
e.g.: quantum dot with barrier at lead

Scattering operator:

$$S(\varepsilon) = R' + T'\left(e^{-i\varepsilon} - FR\right)^{-1} FT$$

$$R = \begin{pmatrix} 1_{1\ldots i} & & \\ & r1_{i\ldots j} & \\ & & 1_{j\ldots M} \end{pmatrix}$$

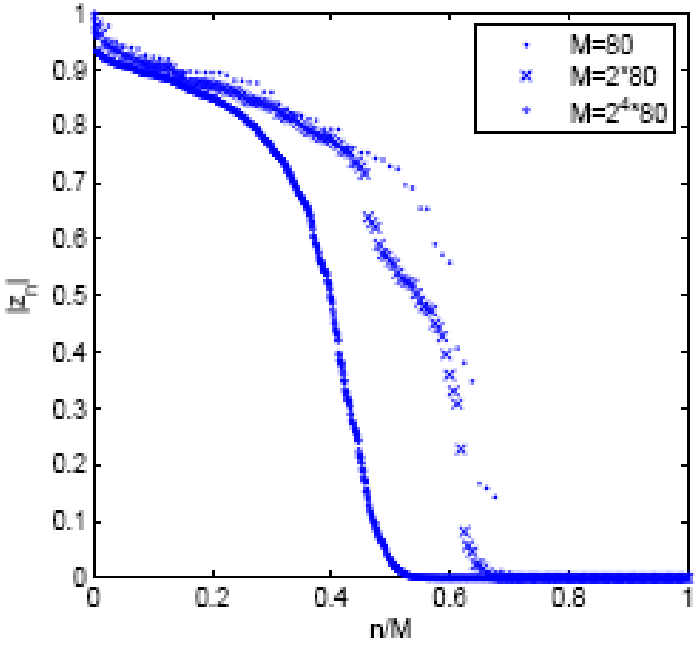
Some qm-cl correspondence remains



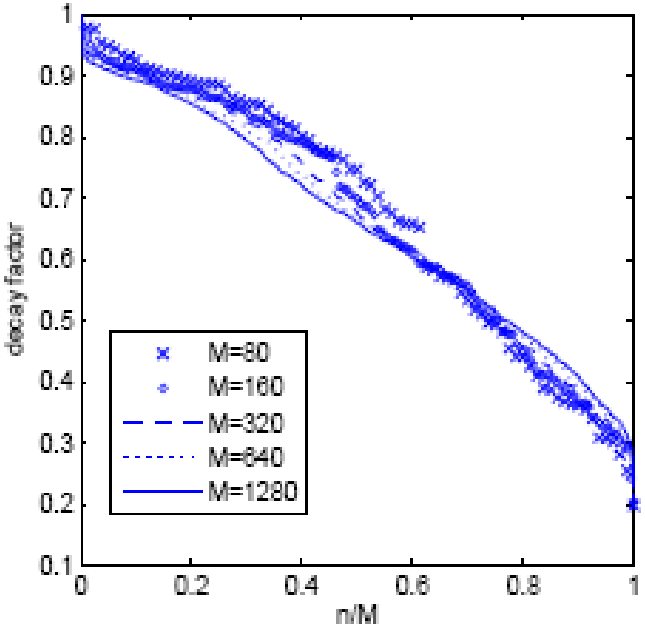
$r^2=0$ $r^2=1/2$

Short-living resonances: life times strongly affected

$r^2=0$

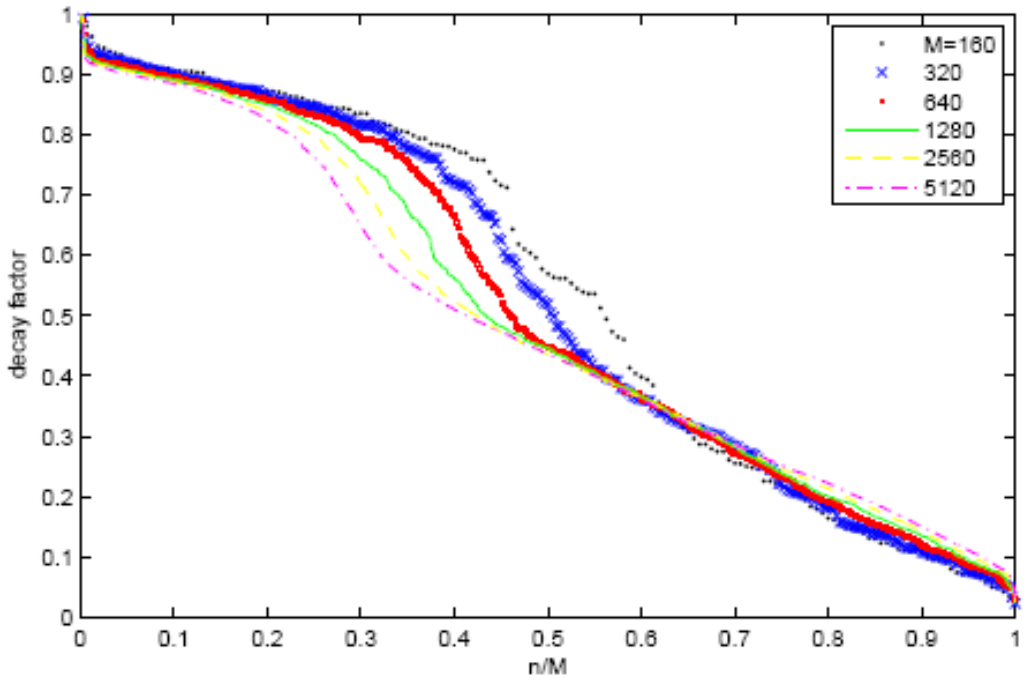


$r^2=1/2$



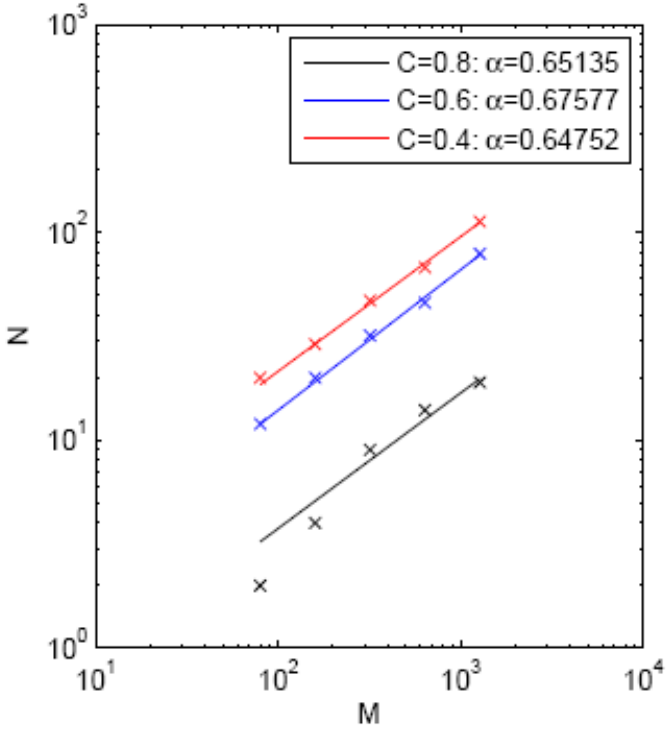
Apparent fractal Weyl law only at small reflection probablity

r^2=0.0001

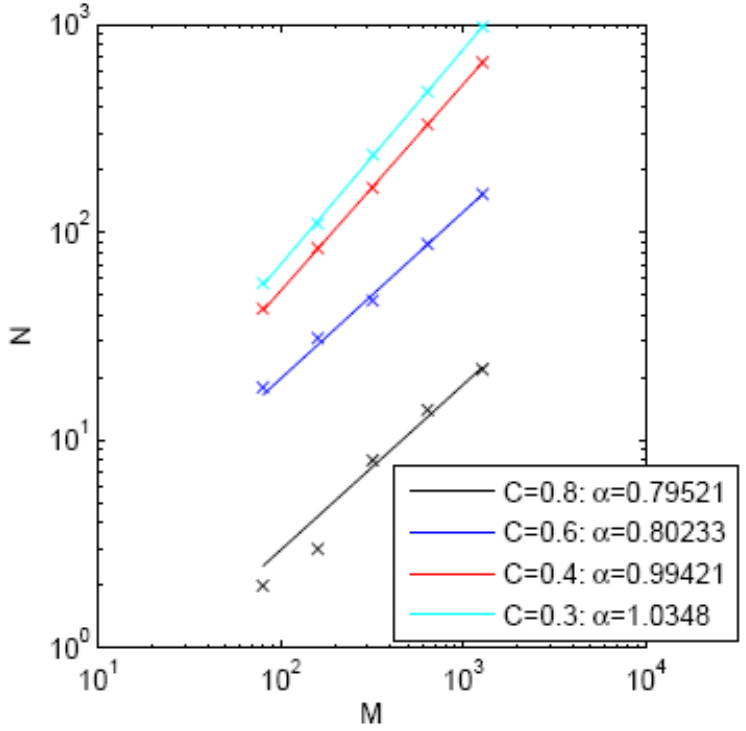


Scaling: cut-off dependent fractal Weyl law even at larger r

$r^2=0$

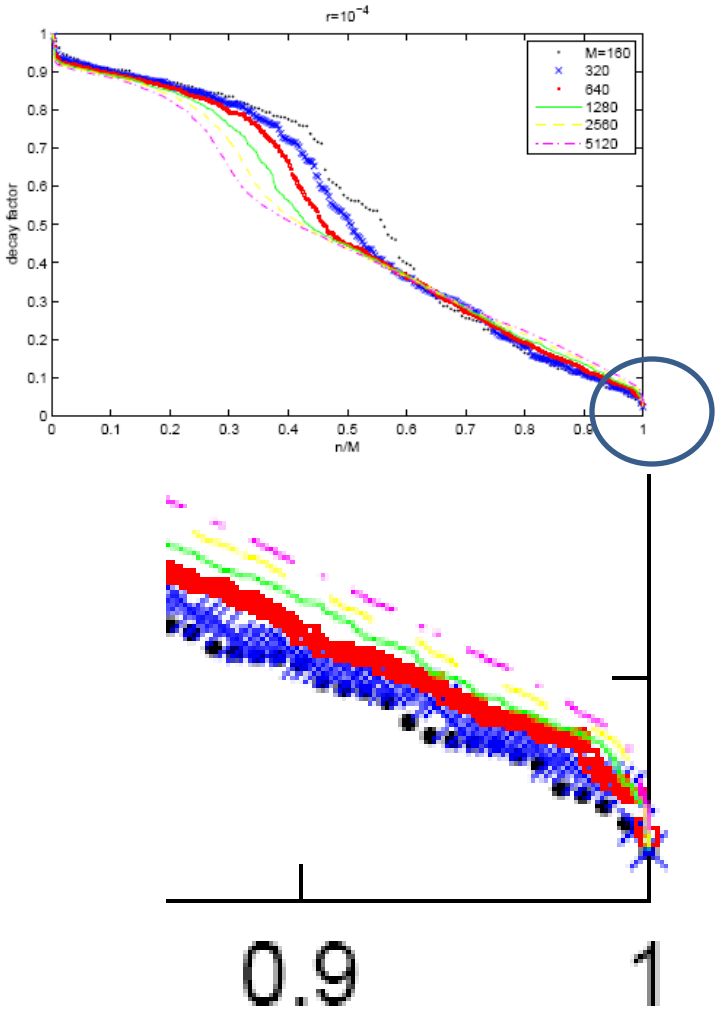


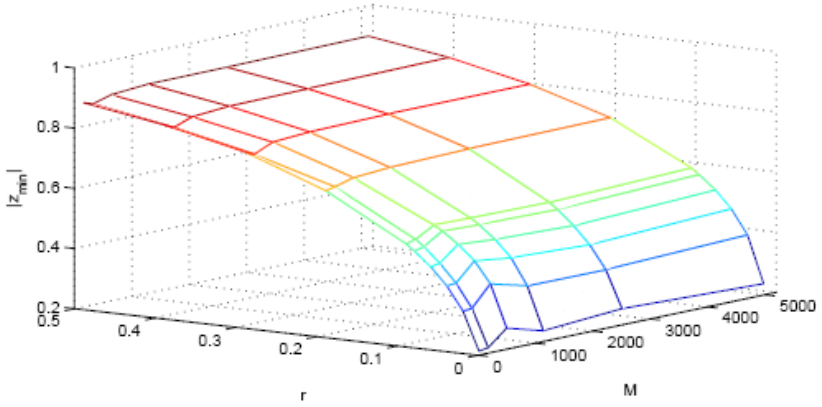
$r^2=0.1$

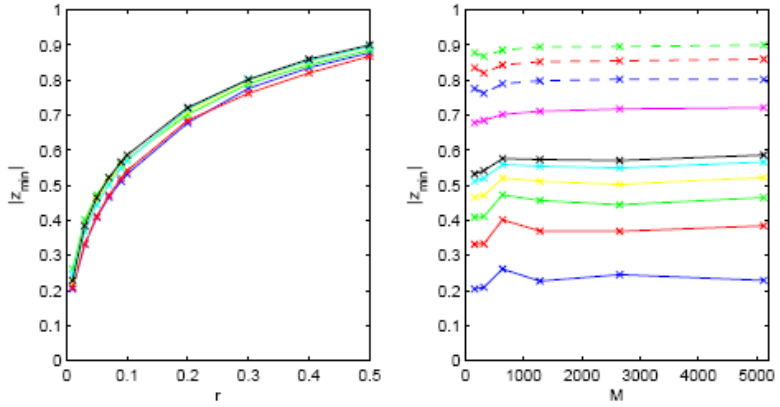


Smallest eigenvalue

r^2=0.0001







independent of k and M:
Universal scaling (like sqrt)

nonballistic escape II: dielectric resonators

Stroboscopic scattering operator

$$S(\omega) = -R + T\left(e^{-i\omega\tau} - FR\right)^{-1} FT$$

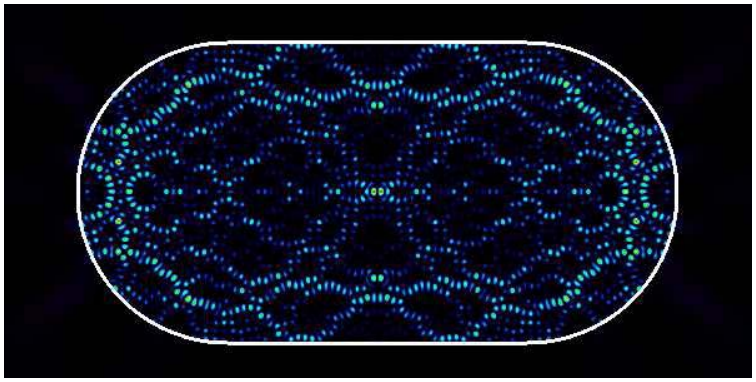
with

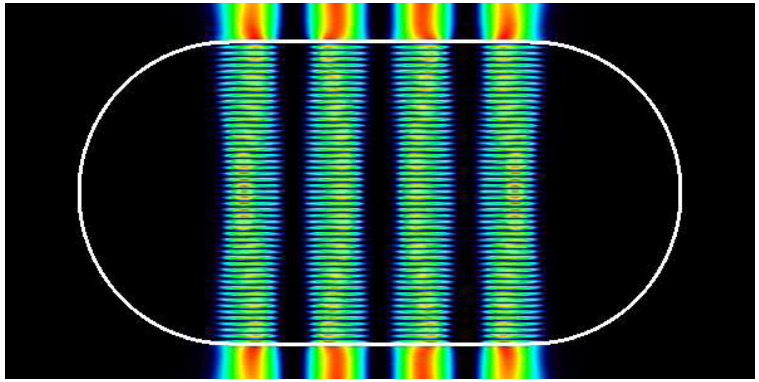
- frequency ω,
- traversal time $\tau = n\,\pi\,A\,/\,v\,C$ (Sabine's law),
- *R*, *T* determined by Fresnel reflection coefficients.

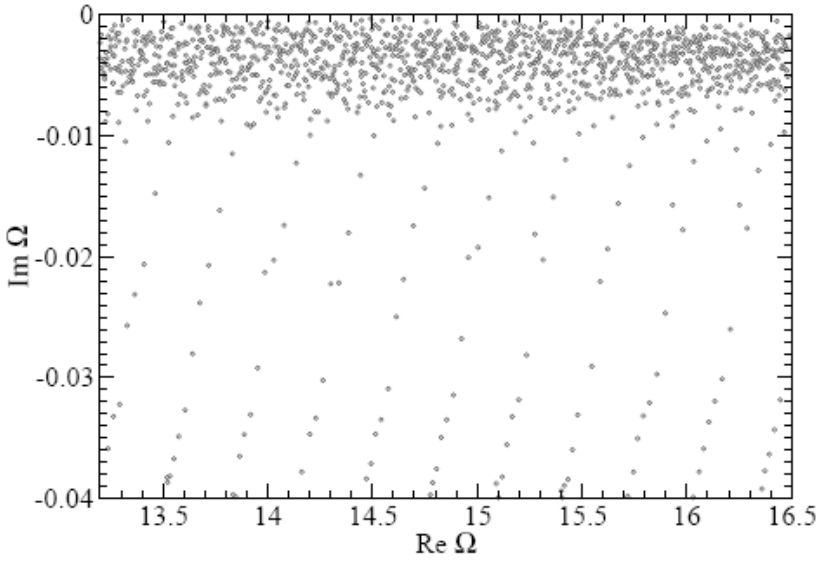
(n: refractive index; A: area, C: perimeter, v: velocity)

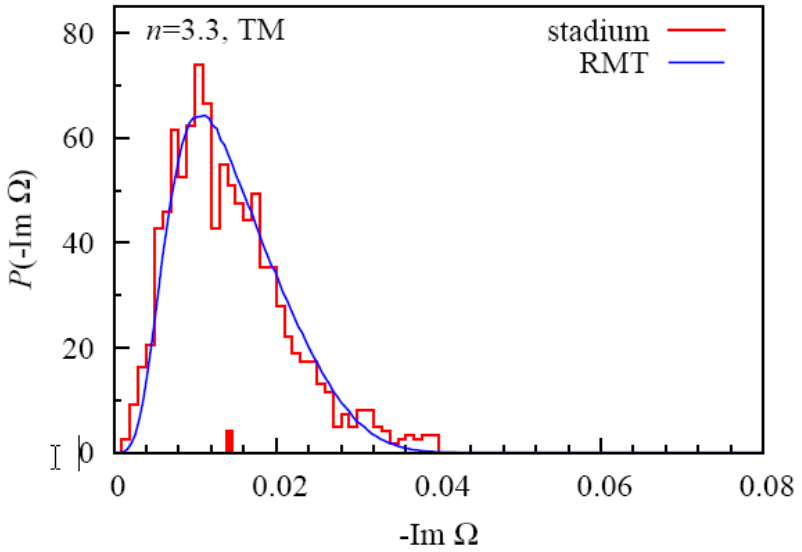
- $M=N=\dim S = \omega\, C/v\, \pi$ (Weyl's law applied to the boundary)

Compare realistic resonator to random matrix theory (RMT)







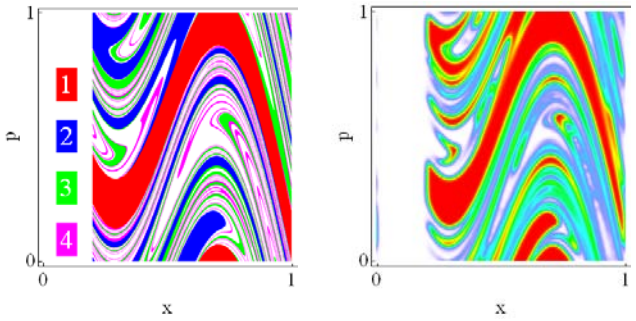


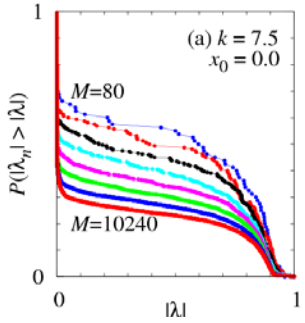
Bands of short-living states (origin: bouncing ball motion)

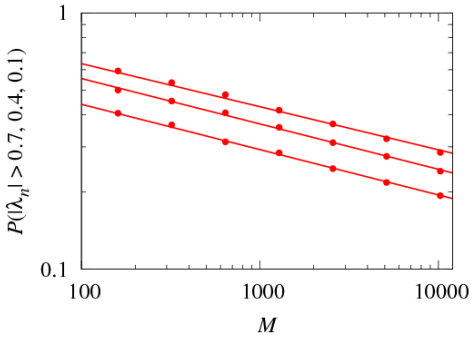
Requires to renormalize *M* and τ! Here done independent from fluctuations by using mean level spacing and decay rate of long-living states.

Summary

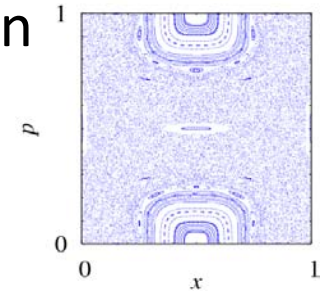
- Phase space rules can be resurrected by semiclassical Schur decomposition; links fractal Weyl law to Ehrenfest time

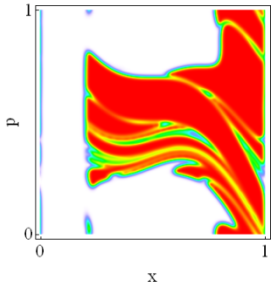


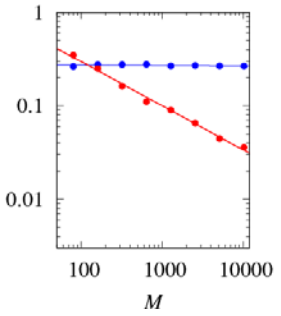




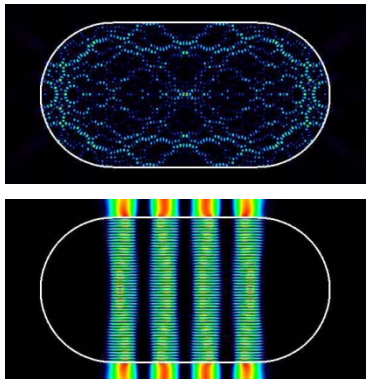
- Fractal Weyl law also exists in generic dynamical systems (mixed phase space)

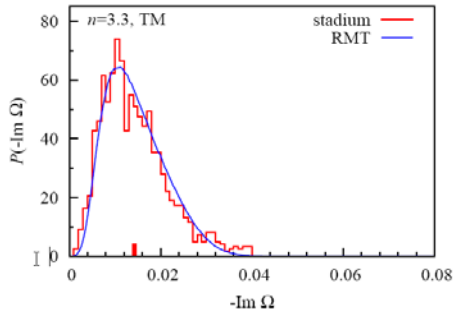






- Non-ballistic decay (q-dots with tunneling, dielectric resonators)





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