Stochastic Bäcklund transformations

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Background

Many recent results relate variations of the Robinson-Schensted-Knuth correspondence, with random input, to quantum integrable systems (Schur processes, Dyson Brownian motion, quantum Toda, etc).

Connections between longest increasing subsequence and last passage percolation problems, random polymers, and random matrix theory.

Variations of Pitman's $2M - X$ theorem.

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Related question: when/how is 'classical + noise = quantum'?

Example

Free particle in one dimension: $H = p^2/2, \quad \dot{x} = p, \quad \dot{p} = 0.$

Constant of motion $p = \lambda$,

$$\dot{x} = \lambda, \qquad x(t) = x(0) + \lambda t.$$

Quantum system: $H = D^2/2$, $D = d/dx$. Eigenfunctions $e^{\lambda x}$,

$$L_\lambda = e^{-\lambda x} \circ (H - \lambda^2/2) \circ e^{\lambda x} = \tfrac{1}{2}D^2 + \lambda D$$

= infinitesimal generator of Brownian motion with drift λ

$$dx = dB + \lambda dt, \qquad x(t) = x(0) + B(t) + \lambda t.$$

Toda

Classical system $\frac{1}{2}p^2 - \frac{1}{2}e^{-2x}$, equation of motion $\ddot{x} = -e^{-2x}$.

Quantum system $H = (\partial_x^2 - e^{-2x})/2$.

Add noise to equations of motion?

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$\longrightarrow$ quantum Toda

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Couple Toda particle to a free particle Bäcklund transformation

Add noise to free particle

$\longrightarrow$ quantum Toda *Stochastic* Bäcklund transformation

Bäcklund transformation

Kernel function $K(x,u) = \exp\big(-e^{-x}\cosh u\big)$ satisfies

$$(\partial_x \ln K)^2 - (\partial_u \ln K)^2 = e^{-2x}, \qquad \partial_x^2 \ln K = \partial_u^2 \ln K.$$

Consider

$$\dot{u} = -\partial_u \ln K = e^{-x}\sinh u, \qquad \dot{x} = \partial_x \ln K = e^{-x}\cosh u.$$

For this coupled system, x satisfies the Toda equation and $\dot{u} = \lambda$ is a conserved quantity. Moreover, λ is an eigenvalue of the Lax matrix

$$\begin{pmatrix} p & e^{-x} \\ -e^{-x} & -p \end{pmatrix}.$$

Bäcklund transformation

The equation $\dot{u} = \lambda$ is equivalent to the (critical point) equation $\partial_u \ln K_\lambda = 0$, where $K_\lambda = e^{\lambda u} K$. Using this equation, namely $\sinh u = \lambda e^x$, we can rewrite the evolution equations as

$$\dot{u} = \lambda, \qquad \dot{x} = \dot{u} + e^{-u-x}.$$

Solution: for $\lambda \in \mathbb{R}$, suppose $\sinh u(0) = \lambda e^{x(0)}$, then

$$u(t) = u(0) + \lambda t, \qquad x(t) = \begin{cases} \ln\left(\frac{1}{\lambda}\sinh u(t)\right) & \lambda \neq 0 \\ \ln\left(e^{x(0)} + t\right) & \lambda = 0. \end{cases}$$

Quantum system

Let $H = (\partial_x^2 - e^{-2x})/2$, and write $H_\lambda = H - \lambda^2/2$. Then

$$H_\lambda \circ K_\lambda = K_\lambda \circ \left(\frac{1}{2}\partial_u^2 + \lambda \partial_u\right).$$

It follows that

$$\psi_\lambda(x) = \int_{-\infty}^{\infty} K_\lambda(x,u)du$$

is an eigenfunction of H with eigenvalue $\lambda^2/2$.

We note that $\psi_\lambda(x) = 2K_\lambda(e^{-x})$, where $K_\nu(z)$ is the modified Bessel function of the second kind, also known as Macdonald's function.

Stochastic system

Define, for suitable $f : \mathbb{R}^2 \to \mathbb{R}$,

$$\tilde{K}_\lambda f(x) = \int_{-\infty}^{\infty} K_\lambda(x,u) f(x,u)\, du,$$

and set

$$A_\lambda = \frac{1}{2}\partial_x^2 + \frac{1}{2}\partial_u^2 + \partial_x\partial_u + \lambda\partial_u + \left(\lambda + e^{-u-x}\right)\partial_x.$$

The above intertwining relation extends to:

$$H_\lambda \circ \tilde{K}_\lambda = \tilde{K}_\lambda \circ A_\lambda$$

This intertwining relation has a *probabilistic* meaning, as follows.

Stochastic Bäcklund transformation

Consider the coupled SDEs obtained by adding white noise to λ in the (rewritten) Bäcklund transformation, that is

$$dU = dB + \lambda dt, \qquad dX = dU + e^{-U-X} dt.$$

This is a diffusion with generator A_λ, and the above intertwining relation yields the following, due to Matsumoto-Yor 99, Baudoin 02:

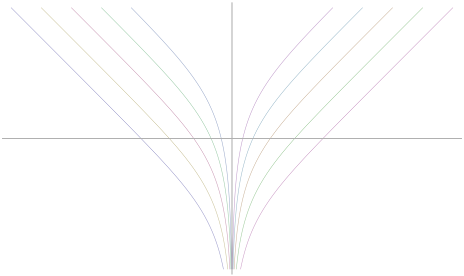
If $X_0 = x$ and $U_0 \sim \psi_\lambda(x)^{-1} K_\lambda(x, u) du$, then X_t is a diffusion process on $\mathbb{R}$ with infinitesimal generator

$$L_\lambda = \psi_\lambda(x)^{-1} H_\lambda \psi_\lambda(x) = \frac{1}{2}\partial_x^2 + \partial_x \ln \psi_\lambda(x) \cdot \partial_x.$$

To summarise, for any given value of the constant of motion $\lambda = \dot{u} \in \mathbb{R}$, the classical flow in $\mathbb{R}^2$ is along the curve $\sinh u = \lambda e^x$, according to

$$\dot{u} = \lambda, \qquad \dot{x} = \dot{u} + e^{-u-x},$$

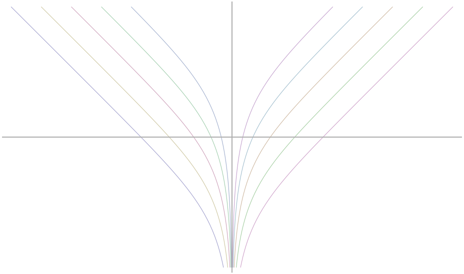
and the x-coordinate satisfies the equation of motion $\ddot{x} = -e^{-2x}$.



If we add noise to the constant of motion λ, the evolution is given by the SDEs

$$dU = dB + \lambda dt, \qquad dX = dU + e^{-U-X} dt$$

and, for appropriate (random) initial conditions, the u-coordinate evolves as a Brownian motion with drift λ and the x-coordinate evolves as a diffusion process in $\mathbb{R}$ with infinitesimal generator L_λ.



Calogero-Moser systems

By choosing different kernel functions, the above construction extends to rational and hyperbolic Calogero-Moser systems, and higher rank systems.

In all cases, the quantum systems (in imaginary time) are obtained by adding noise to the constants of motion in particular constructions of classical systems based on Bäcklund transformations.

For example, for rational CM with Hamiltonian $\frac{1}{2}(p^2 - \mu^2/x^2)$ can take

$$K(x,u) = (x - u^2/x)^\mu, \qquad |u| \leq x.$$

The case $\mu = 0$ corresponds to Pitman's $2M - X$ theorem.

cf. Dyson's Brownian motion

KPZ equation and semi-infinite Toda chain

Stochastic heat equation (SHE)

$$u_t = \tfrac{1}{2} u_{xx} + \xi u$$

where $\xi(t, x)$ is space-time white noise. Related to KPZ equation

$$h_t = \tfrac{1}{2} h_{xx} + \tfrac{1}{2} (h_x)^2 + \xi$$

via Cole-Hopf transform $h = \log u$.

KPZ equation and semi-infinite Toda chain

Multilayer extension (O'C-Warren 11): start with solution $u(t,x,y)$ to SHE with $u(0,x,y) = \delta(x-y)$ and define τ_n by chaos expansions which are formally

$$\tau_n = \det[\partial_x^{i-1}\partial_y^{j-1}u]_{i,j=1,\dots,n}.$$

Evolution described (again formally) by the coupled equations

$$\partial_t a_n = \frac{1}{2}\partial_x^2 a_n + \partial_x[a_n \partial_x h_n]$$

where $a_n = \tau_{n-1}\tau_{n+1}/\tau_n^2$ and $h_n = \log(\tau_n/\tau_{n-1})$ with the convention $\tau_0 = 1$.

This gives a continuum version version of geometric RSK and should be thought of as a stochastic Bäcklund transformation for the 'semi-infinite quantum Toda chain'.

KPZ equation and semi-infinite Toda chain

If we switch off the noise by setting $\xi = 0$, then u is given by the heat kernel

$$u(t,x,y) = \frac{1}{\sqrt{2\pi t}} e^{-(x-y)^2/2t}$$

and

$$\tau_n = t^{-n(n-1)/2} \left(\prod_{j=1}^{n-1} j! \right) u^n.$$

Then $(\ln \tau_n)_{xx} = -a_n$ or, equivalently,

$$(h_n)_{xx} = e^{h_n - h_{n-1}} - e^{h_{n+1} - h_n}$$

(with $h_0 \equiv +\infty$) which are the equations of motion of the semi-infinite Toda chain.