Non-unitary maps, random matrices and convolutions of Marchenko–Pastur distribution

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in collaboration with

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Random Matrices, Brunel University, Dec. 13, 2014

A) Random density matrices

Mixed quantum state = density operator which is

a) **Hermitian**, $\rho = \rho^*$,
b) **positive**, $\rho \geq 0$,
c) **normalized**, $\mathrm{Tr}\rho = 1$.

Let $\mathcal{M}_N$ denote the set of density operators of size N.

Ensembles of random states in $\mathcal{M}_N$

Random matrix theory point of view:
Let A be matrix from an arbitrary **ensemble** of **random matrices**.
Then

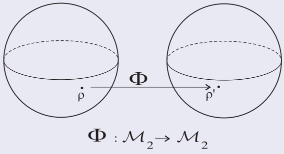
$$\rho \;=\; \frac{AA^*}{\mathrm{Tr}AA^*}$$

forms a **random quantum state**

fixed trace ensembles : (Hilbert–Schmidt, Bures, etc.)
Hans–Jürgen Sommers, K.Ż, (2001, 2003, 2004, 2010)

B) Quantum Maps & Nonunitary Dynamics

Quantum operation: a linear, completely positive trace preserving map Φ acting on a density matrix ρ



positivity: $\Phi(\rho) \geq 0, \quad \forall \rho \in \mathcal{M}_N$

complete positivity: $[\Phi \otimes \mathbb{1}_K](\sigma) \geq 0, \quad \forall \sigma \in \mathcal{M}_{KN}$ and $K = 2, 3, ...$

The Kraus form

$\rho' = \Phi(\rho) = \sum_i A_i \rho A_i^*$, where the Kraus operators satisfy $\sum_i A_i^* A_i = \mathbb{1}$, which implies that the trace is preserved

allows one to represent the **superoperator Φ** as a matrix of size N^2

$$\Phi = \sum_i A_i \otimes \bar{A}_i$$

Interacting quantum dynamical systems

Generalized **quantum baker map** with measurements

a) Quantisation of **Balazs and Voros** applied for the asymmetric map

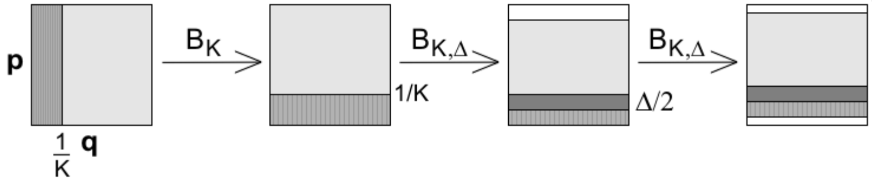
$$B = F_N^* \begin{bmatrix} F_{N/K} & 0 \\ 0 & F_{N(K-1)/K} \end{bmatrix}, \quad \text{where} \quad N/K \in \mathbb{N}.$$

where F_N denotes the **Fourier matrix** of size N. Then $\rho' = B\rho_i B^*$

b) M **measurement operators** projecting into orthogonal subspaces

Kraus form: $\rho_{i+1} = \sum_{i=1}^{M} P_i \rho' P_i$

c) vertical **shift** by $\Delta/2$ (**Łoziński, Pakoński, Życzkowski** 2004)

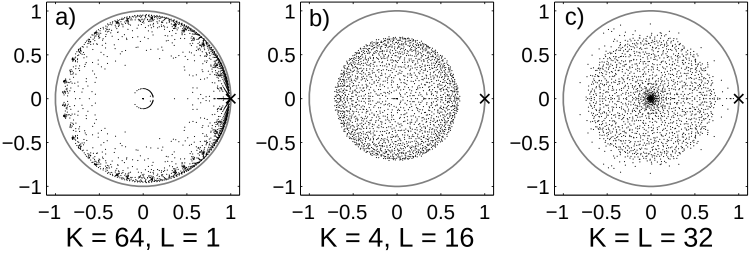


Standard classical model $K = 2$, **dynamical entropy** $H = \ln 2$;

Asymmetric model, $K > 2$, entropy decreases to zero as $K \to \infty$.

Classical limit: $N \to \infty$ with $K \le N$.

Exemplary spectra of superoperator for L–fold generalized baker map B^L & measurement with M **Kraus operators** for $N = 64$ and $M = 2$:



a) weak chaos ($K = 64$ and $L = 1$),

b) **strong chaos** ($K = 4$ and $L = 4$) – **'universal' behaviour**:

$\lambda_1 = 1$ and **uniform Girko disk of eigenvalues** of radius R,

(described by **real Ginibre** ensemble).

c) weak chaos ($K = 32$ and $L = 32$).

Conjecture

on spectral properties of superoperators describing non–unitary dynamics

In the case of

strong chaos and

large interaction with the environment

the superoperators display **'universal' behaviour** and can be described by random matrices pertaining to **real Ginibre** ensemble.

Real ensemble is used since the map preserves hermicity of density matrix ρ and superoperator Φ can be represented by a **real** matrix.

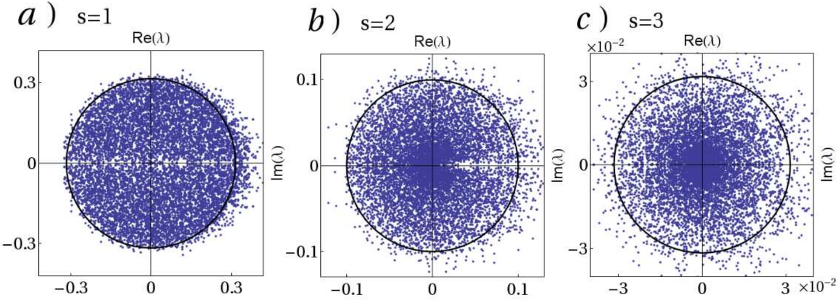
For instance, in the **Bloch vector** τ representation any state of size N is given by $\rho = \mathbb{1}/N + \tau \cdot \lambda$ and the map $\rho' = \Phi(\rho)$ reads

$$\tau' = \Phi\tau$$

where Φ is a real matrix of size $N^2 - 1$ and τ a generalized **Bloch vector** of length $N^2 - 1$, while λ is a vector of generators of $SU(N)$ with $N^2 - 1$ components.

s–steps propagators for perturbed baker map Φ_B^s

Exemplary spectra of superoperator Φ_B^s for s–**steps** non-unitary evolution



i) spectral properties of 1–**step** propagator Φ coincide with these of real random Ginibre matrices (uniform disk apart of the real axis)

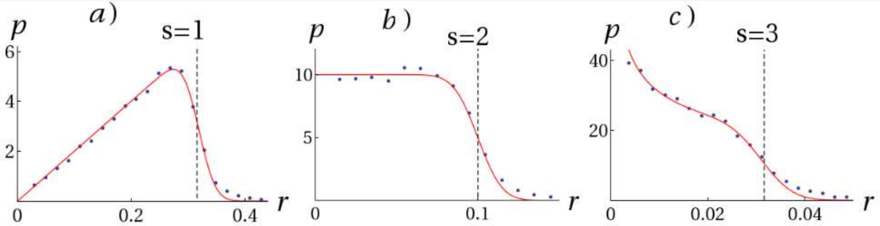
ii) properties of s–**step** propagators Φ^s are similar as products of random matrices:

a) the radial density of complex eigenvalues $r = |z|$ of Φ_B^s

of the periodically measured baker map displays asymptotically an **algebraic power law** for **products** of s random Ginibre matrices

of **Burda, Jarosz, Livan, Nowak, Święch, 2010**:

$$P_s(r) \sim r^{-1+2/s}$$



with an error-function Ansatz (red line) describing the **finite** N effects.

b) the squared singular values of Φ_B^s for non-unitary baker map

can be described by **Fuss-Catalan distribution** of order $t = s - 1$.

Let $x = N^2 \lambda$, where λ is an eigenvalue of $\Phi^s(\Phi^s)^*$. Then

$s = 2$, $t = 1$ (Wishart)

$P_1(x) = \frac{\sqrt{1-x/4}}{\pi\sqrt{x}} \quad x \in [0, 4]$,

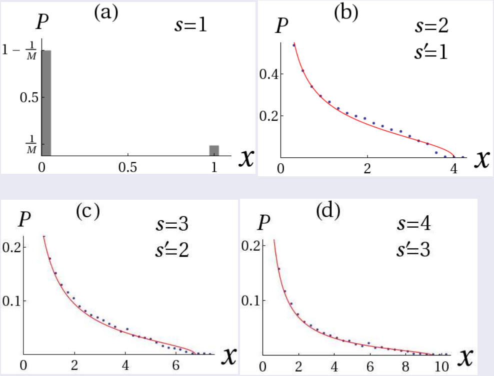
Marchenko–Pastur distrib.

(with moments given by the **Catalan** numbers);

$s \geq 3$, $t \geq 2$, the **Fuss–Catalan** distrib. $P_t(x)$ for $x \in [0, (t+1)^{t+1}/t^t]$

(with moments given by the **Fuss–Catalan** numbers) expicitely derived in

Penson, K. Ż., 2011



Fuss-Catalan numbers $FC_s(n)$

Generalized Fuss-Catalan numbers are defined for any integer s,

$$FC_s(n) \; := \; \frac{1}{sn+1}\binom{sn+n}{n}$$

some examples of FC numbers:

$$\begin{aligned}
FC_1(n) &= 1, 1, 2, 5, 14, 42, 132, 427, \ldots \\
FC_2(n) &= 1, 1, 3, 12, 55, 273, 1428, 7752, \ldots \\
FC_3(n) &= 1, 1, 4, 22, 140, 969, 7084, 53820, \ldots \\
FC_4(n) &= 1, 1, 5, 35, 285, 2530, 23751, 231880, \ldots \; .
\end{aligned}$$

where $n = 0, \ldots, 7$.

The family $FC_1(n)$ coincides with the standard **Catalan** numbers, and gives the moments of the **Marchenko–Pastur** distribution.

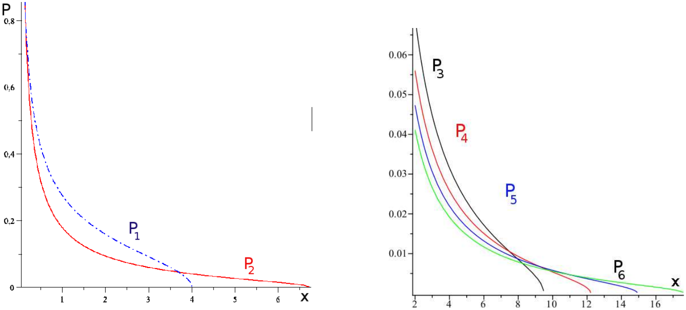
Fuss-Catalan distribution P_s

defined for an integer number s is characterized by its **moments**

$$\int x^n P_s(x)dx = \frac{1}{sn+1}\binom{sn+n}{n} = FC_s(n)$$

equal to the **generalized Fuss-Catalan numbers** .

The density P_s is analitic on the support $[0, (s+1)^{s+1}/s^s]$, while for $x \to 0$ it behaves as $1/(\pi x^{s/(s+1)})$.



The same moments decribe (asymptotically) distribution of singular values for s–th power of Ginibre G^s, (**Alexeev, Götze, Tikhomirov** 2010)

Fuss-Catalan distributions P_s

The moments of P_s are equal to Fuss-Catalan numbers.
Using inverse **Mellin transform** one can represent P_s by the **Meijer** G**–function**, which in this case reduces to s **hypergeometric functions**

Exact explicit expressions for FC P_s

$s=1$, $P_1(x) = \frac{1}{\pi\sqrt{x}}\ {}_1F_0\left(-\frac{1}{2};\ ;\ \frac{1}{4}x\right) = \frac{\sqrt{1-x/4}}{\pi\sqrt{x}}$, **Marchenko–Pastur**

$$s=2,\ P_2(x) = \frac{\sqrt{3}}{2\pi x^{2/3}}\ {}_2F_1\left(-\frac{1}{6},\frac{1}{3};\ \frac{2}{3};\ \frac{4x}{27}\right) - \frac{\sqrt{3}}{6\pi x^{1/3}}\ {}_2F_1\left(\frac{1}{6},\frac{2}{3};\ \frac{4}{3};\ \frac{4x}{27}\right) =$$

$$= \frac{\sqrt[3]{2}\sqrt{3}}{12\pi}\ \frac{\sqrt[3]{2}\left(27+3\sqrt{81-12x}\right)^{\frac{2}{3}} - 6\sqrt[3]{x}}{x^{\frac{2}{3}}\left(27+3\sqrt{81-12x}\right)^{\frac{1}{3}}}$$ **Fuss–Catalan**

Arbitrary s, $\Rightarrow$ $p_s(x)$ is a superposition of
s **hypergeometric functions,**

$$P_s(x) = \sum_{j=1}^{s} \beta_j\ {}_sF_{s-1}\left(a_1^{(j)},\dots,a_s^{(j)};\ b_1^{(j)},\dots,b_{s-1}^{(j)};\ \alpha_j x\right)\ .$$

Penson, K. Ż., *Phys. Rev.* **E** 2011.

Raney numbers

Generalisation of FC numbers: the **Raney numbers**

$R_{p,r}(n) := \frac{r}{pn+r}\binom{pn+r}{n}$

defined for $n = 0, 1, \ldots$ and related in combinatorics to **Raney sequences**.

In a particular case $R_{s+1,1}(n)$ are equal to $FC_s(n)$.

By definition the following property holds $R_{p,p}(n) = R_{p,1}(n+1)$.

examples of Raney numbers:

$$
\begin{aligned}
R_{4,2}(n) &= 1, 2, 9, 52, 340, 2394, 17710, 135720, ...(A069271) \\
R_{4,5}(n) &= 1, 5, 30, 200, 1425, 10626, 81900, 647280, ... , \\
R_{5,2}(n) &= 1, 2, 11, 80, 665, 5980, 56637, 556512, ...(A118969), \\
R_{6,3}(n) &= 1, 3, 21, 190, 1950, 21576, 250971, 3025308, ...
\end{aligned}
$$

where $n = 0, \ldots, 7$.

Raney distributions

Result of **Młotkowski** (2011):

If a point (r, p) satisfies inequalities $p \geq 1$ and $0 < r \leq p$
the **Raney numbers** $R_{p,r}(n)$ describe the moments of a **measure** $\mu_{p,r}$
with a compact support.

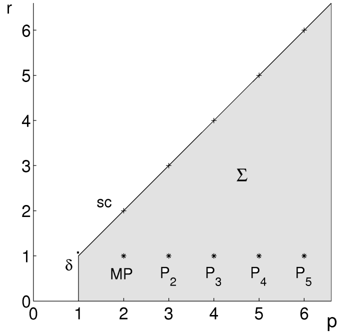
Point $(1, 1)$ implies constant moments, $R_{1,1}(n) = 1$,
which represents a singular, Dirac delta measure, $\mu_{1,1} = \delta(x - 1)$.

If a measure $\mu_{p,r}$ is represented by a density we will denote it by $W_{p,r}(x)$.

$$\int x^n W_{p,r}(x) dx = R_{p,r}(n) = \frac{r}{pn+r}\binom{pn+r}{n}$$

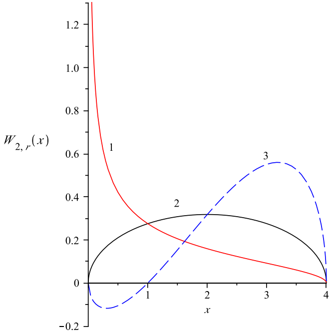
Then $W_{s+1,1}(x)$ reduces to the **Fuss–Catalan** probability density $P_s(x)$.

Raney distributions II



Set Σ of positive measures in the parameter space (p, r)

Młotkowski (2010)

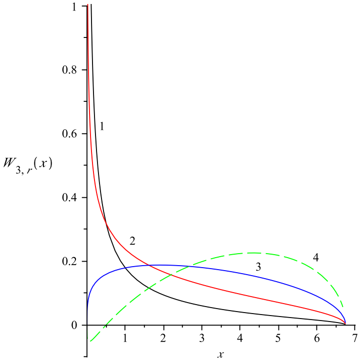


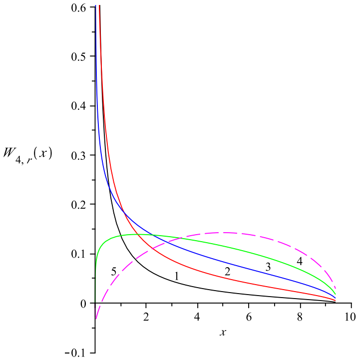
Example: **Raney distribution** $W_{2,2}(x)$ gives the **semicircle law** centered at $x = 2$ with support $x \in [0, 4]$

$W_{2,1}(x)$ is **Marchenko-Pastur** while $W_{3,1}(x)$ is not **positive** !

Raney distributions III

exemplary **Raney distributions** $W_{p,r}(x)$





Raney distribution $W_{3,r}(x)$ are positive for $r = 1, 2, 3$ and supported in $x \in [0, 6\frac{2}{3}]$

Raney distribution $W_{4,r}(x)$ are positive for $r = 1, 2, 3, 4$.

Raney distributions IV

Some properties:

- For integer values of p, q the function $W_{p,r}(x)$ is supported in the interval $[0, p^p/(p-1)^{p-1}]$
- The distribution $W_{p,r}(x)$ is non-negative if $1 \leq r \leq p$
- For $p > r$ the distribution $W_{p,r}(x)$ diplays for small x a singularity of the type $x^{-(p-r)/p}$.
- For $p = r$ the 'diagonal' Raney distributions behave for $x << 1$ as $W_{p,p}(x) \sim x^{1/p}$.
- The folowing relation holds (**Młotkowski 2010)** $W_{p,p}(x) = x \, W_{p,1}(x) = x \, \pi^{p-1}(x)$

Exemplary formula for an intermediate **Raney distribution**

$$W_{3,2}(x) = \frac{\sqrt{3}\sqrt[3]{2}}{36\pi} \, x^{-\frac{1}{3}} \, \frac{\left[\left(27+3\sqrt{81-12x}\right)^{\frac{4}{3}} - 18\sqrt[3]{2}x^{\frac{2}{3}}\right]}{\left(27+3\sqrt{81-12x}\right)^{\frac{2}{3}}}$$

St. Mary Church, **Cracow, Poland**

Multiplicative convolutions and Cauchy transform

Consider S–transform of **Voiculescu**, $S(w) = \frac{1+w}{w}\chi(w)$, where

$$\frac{1}{\chi(w)} G\left(\frac{1}{\chi(w)}\right) - 1 = w.$$

To recover the resolvent we put $\frac{1}{\chi(w)} \ = \ z$
and for a random matrix ensemble M write the **Cauchy function**

$$G(z) \equiv \frac{1}{N} \left\langle \mathrm{tr} \frac{1}{z\mathbf{1}_N - M} \right\rangle = \frac{1+w(z)}{z}, \qquad (*)$$

as its imaginary part yields the spectral function

$$\rho(\lambda) = -\frac{1}{\pi} \lim_{\epsilon \to 0} \Im G(z)|_{z=\lambda+i\epsilon}. \qquad (**)$$

To derive it we solve the algebraic equation

$$zw(z)\ S\big(w(z)\big) = 1 + w(z) \qquad (***)$$

with respect to w and plug $w(z)$ into (*) and evaluate (**).

Example: S–transform of **Marchenko–Pastur**, $S_{MP}(w) = \frac{1}{1+w}$
leads to a **quadratic** equation, $zw = (1+w)^2$ for $w = w(z)$
which yields the MP distribution, $P_{\mathrm{MP}}(x) = P_1(x) = \frac{1}{2\pi}\sqrt{\frac{4}{x} - 1}$.

Quadratic equations, $w(z) = a_2 z^2 + a_1 z + a_0 = 0$

i) 'Quadratic' distributions for density of $AA^*/\mathrm{Tr}AA^*$, where A is

a) rectangular Ginibre matrices with **rectangularity** $c = K/N$, described by S–transform, $S_{MP,c}(w) = 1/(1+cw)$

leads to $zw = (1+w)(1+cw)$ $\quad (***)$

which yields the general **Marchenko–Pastur** distribution,

$$P_{1,c}(x) = \frac{1}{2\pi x c}\sqrt{(x-x_-)(x_+-x)} \quad \text{with } x_\pm = 1 + c \pm 2\sqrt{c}.$$

b) sum of two **Haar** random unitary matrices, $A = U_1 + U_2$. Then the S-transform $S_{AS}(z) = (z+2)/(2+2z)$

leads to equation $wz(w+2) = 2(1+w)^2$ $\quad (***)'$

which yields the **Arcsine distribution**,

$$AS(x) = \frac{1}{\pi}\frac{1}{\sqrt{x(2-x)}}, \quad x \in [0,2].$$

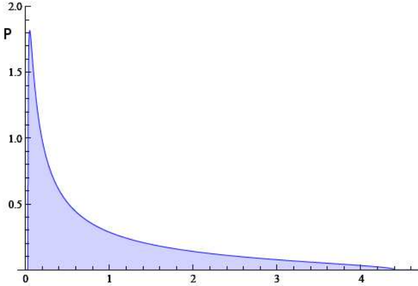
ii) 'Cardano qubic' distributions for density of $AA^*/\mathrm{Tr}AA^*$, where A is a) product of two square Ginibre matrices,

$$S_{FC_2}(w) = [S_{MP}(w)]^2 = 1/(1+w)^2$$

gives equation $wz = (1+w)^3$ and **Fuss-Catalan** distribution,

$$P_2(x) = [P_1(x)]^{\boxtimes 2} = \frac{\sqrt[3]{2}\sqrt{3}}{12\pi} \frac{\sqrt[3]{2}\left(27+3\sqrt{81-12x}\right)^{\frac{2}{3}} - 6\sqrt[3]{x}}{x^{\frac{2}{3}}\left(27+3\sqrt{81-12x}\right)^{\frac{1}{3}}}.$$

a') product of two rectangular Ginibre matrices,
with **rectangularity** $c = K/N$
leads to $S_{2,c} = 1/(1+cw)^2$
so $wz = (1+w)(1+cw)^2$.
This gives the generalized **Fuss-Catalan** distribution, $P_{2,c}(x)$, plotted for $c = 1/2$.



Qubic equations, $w(z) = a_3 z^3 + a_2 z^2 + a_1 z + a_0 = 0$

ii) 'Cardano qubic' distributions:

b) **Free square root** of MP, $P_{1/2}(x) = [P_1(x)]^{\boxtimes 1/2}$

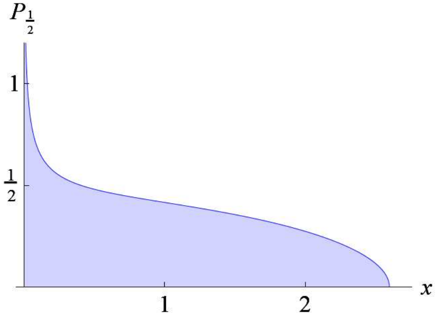
corresponds to $[S_{MP}(w)]^{1/2} = 1/\sqrt{1+w}$.

Solution of $w^3 + (3 - z^2)w^2 + 3w + 1 = 0$ yields

$$P_{1/2}(x) = x^{-1/3}\frac{(9+Y)^{1/3} - (9-Y)^{1/3}}{2^{4/3}3^{1/6}\pi} + x^{1/3}\frac{(9+Y)^{2/3} - (9-Y)^{2/3}}{2^{4/3}3^{5/6}\pi}$$

where $Y(x) = \sqrt{81 - 12x^2}$ and $x \in [0, \sqrt{27/4}]$.

Free **square root** of **Marchenko–Pastur** distribution, $P_{1/2}(x) = [P_1(x)]^{\boxtimes 1/2}$.



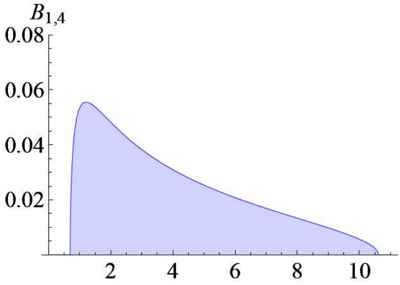
Qubic equations, $w(z) = a_3 z^3 + a_2 z^2 + a_1 z + a_0 = 0$

ii) 'Cardano qubic' distributions for density of $\rho = AA^*/\mathrm{Tr}AA^*$, where $A = (U_1 + U_2)G$ so that $S_{B_1}(w) = \frac{w+2}{2w+2} \cdot \frac{1}{1+w}$ gives equation $wz(w+2) = 2(1+w)^3$ and **Bures** distribution,

$$B_1(x) = \frac{1}{4\pi\sqrt{3}} \left[\left(\frac{a}{x} + \sqrt{\left(\frac{a}{x}\right)^2 - 1} \right)^{2/3} - \left(\frac{a}{x} - \sqrt{\left(\frac{a}{x}\right)^2 - 1} \right)^{2/3} \right],$$

with $a = 3\sqrt{3}$.

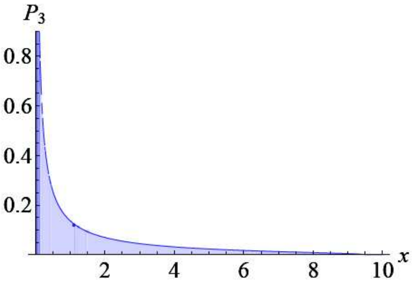
c') rectangular Ginibre matrices, with **rectangularity** $c = K/N$ leads to $S_{B_{1,c}}(w) = (w+2)/(2(1+w)(1+cw))$ so $wz(w+2) = 2(1+cw)(1+w)^2$. This gives the generalized **Bures distribution** $B_{1,c}(x) = AS \boxtimes P_{1,c}$ plotted for $c = 4$.



Quartic equations, $w(z) = a_4 z^4 + a_3 z^3 + a_2 z^2 + a_1 z + a_0$

iii) 'Farrari quartic' distributions for level density of $\rho = AA^*/\mathrm{Tr}AA^*$,
where a) $A = G_1 * G_2 * G_3$
so that $S_3(w) = S_{MP}^3 = 1/(1+w)^3$
gives a quartic equation $w^4 + 4w^3 + 6w^2 + w(4-z) + 1 = 0$
and **Fuss–Catalan** distribution, $P_3(x) = \frac{x^{-3/4}}{2 \cdot 3^{1/4} \pi} \sqrt{4Y - \frac{3^{3/4} x^{1/4}}{\sqrt{Y}}}$,
where $Y(x) = \cos\left[\frac{1}{3} \arccos\left(\frac{3\sqrt{3}}{16}\sqrt{x}\right)\right]$ and $x \in [0, 256/27]$.

Fuss-Catalan distribution
of order $s = 3$,
$P_3(x) = [P_1(x)]^{\boxtimes 3}$



Quartic equations, $w(z) = a_4 z^4 + a_3 z^3 + a_2 z^2 + a_1 z + a_0$

iii) 'Farrari quartic' distributions – other cases studied:

b) **Free third root** of MP, $P_{1/3}(x) = [P_1(x)]^{\boxtimes 1/3}$ based on $w^4 + (4 - z^3)w^3 + 6w^2 + 4w + 1 = 0$ leads to

$P_{1/3}(x) = \frac{1}{2\pi x}\left[Y + 4x^3 - \frac{1}{2}x^6 + \left| \frac{x^3(24-12x^3+x^6)}{4\sqrt{Y-2x^3+\frac{1}{4}x^6}} \right| \right]^{1/2}$, where

$Y(x) = (4/\sqrt{3})x^{3/2} \cos\left[\frac{1}{3} \arccos\left(\frac{3\sqrt{3}}{16} x^{3/2} \right) \right]$ and $x \in [0, (256/27)^{1/3}]$.

c) **2-Bures distributions**, $A = (U_1 + U_2)G_1 G_2$ so that $B_2 = AS \boxtimes P_2$ is the free convolution of arcsine and the 2–Fuss Catalan distribution,

$$B_2(x) = \frac{1}{\pi\, 2^{5/4} x^{3/4}} \sqrt{2 - \sqrt{x/2}}$$

d) generalization for rectangular Ginibre matrices G_1, G_2 with **rectangularity** $c = K/N$ leads to **generalized 2-Bures distributions**, $B_{2,c} = AS \boxtimes P_{2,c}$.

more details in **W. Młotkowski, M.A. Nowak, K. Penson, K.Życzkowski**, preprint **arXiv: 1407.1282**

Recent related works

on products of random matrices, **Fuss–Catalan** and **Raney** distributions, generalizations of Bures distribution, Kesten distributions, Mellin transforms and Meier G–functions, ...

- A. Jarosz (2012)
- G. Akemann, M. Kieburg, L. Wei (2013); G. Akemann, J.R. Ipsen and M. Kieburg (2013); G. Akemann, Z. Burda (2014)
- U. Haagerup and S. Möller (2013)
- A. Kuijlaars and L. Zhang (2013); A. Kujlaars, D. Stivingy (2014)
- P. Forrester, D.-Z. Liu, (2014); P. Forrester, D.-Z. Liu, P. Zinn-Justin (2014); P. J. Forrester and M. Kieburg (2014)
- T. Dupic and I. P. Castillo (2014),
- J.-G. Liu and R. Pego (2014)
- O. Arizmendi and T. Hasebe (2014)
- T. Neuschel, (2014); T. Neuschel, D. Stivingy (2014); W. Gawronski, T. Neuschel, D. Stivigny (2014),

see also posters by **Jesper Ipsen** and **Thorsten Neuschel**

Concluding Remarks

- **Random mixed state** of size N from the **induced ensemble** (which leads to **Marchenko-Pastur** spectral density) is obtained by the partial trace of a composite system in an initially random pure state.
- Other ensembles of random pure states + **partial trace** generate **states** with **Arcsine, k–Kesten, Bures, s–Fuss-Catalan** distribustions.
- **Real Ginibre** matrices are applicable to describe superoperators associated with quantum maps under the condition of **strong chaos** and **large interaction** with the environment
- Explicit and exact analytical forms of the **Fuss-Catalan** and **Raney** distributions expressed as combinations of **hypergeometric** functions. In cases related to equations of order $2, 3$ and 4 the **Cauchy transform** leads to elementary representation for a wide class of models of random matrices.

See you in **Cracow !**