by applying the first algorithm. The resulting trajectories from the vertices of X_0 are shown in Fig. 1. The control sequences for the initial states v_1 and v_2 are shown in Fig. 2.

VI. Conclusion

In this paper, the CRP with an *a priori* given set of initial states has been investigated. In the first part of the paper, an answer to the open problem of deriving necessary and sufficient conditions for the existence of a solution to this problem has been given. Then three techniques for the derivation of such a solution have been presented. The proposed techniques will always provide a solution to this problem if one exists. It should be noted that the necessary and sufficient conditions established in this paper extend a result reported in [10] where sufficient existence conditions were developed. The design techniques established in this paper establish a solution even if there does not exist a control law making the set of initial states positively invariant.

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Robust H_2/H_∞-State Estimation for Systems with Error Variance Constraints: The Continuous-Time Case

Zidong Wang and H. Unbehauen

Abstract—This paper is concerned with the state estimator design problem for perturbed linear continuous-time systems with H_∞ norm and variance constraints. The perturbation is assumed to be time-invariant and norm-bounded and enters into both the state and measurement matrices. The problem we address is to design a linear state estimator such that, for all admissible measurable perturbations, the variance of the estimation error of each state is not more than the individual prespecified value, and the transfer function from disturbances to error state outputs satisfies the prespecified H_∞ norm upper bound constraint, simultaneously. Existence conditions of the desired estimators are derived in terms of Riccati-type matrix inequalities, and the analytical expression of these estimators is also presented. A numerical example is provided to show the directness and effectiveness of the proposed design approach.

***Index Terms*— Algebraic matrix inequality, H_∞ state estimation, Kalman filtering, perturbed systems, robust state estimation.**

I. Introduction

The last three decades have witnessed significant advances in the celebrated Kalman filtering which seems to be the most effective estimation approach; see [1]. This filtering approach assumes that the system model under consideration is exactly known and its disturbances are Gaussian noises with known statistics. However, it is a well-known fact that the design of an optimal Kalman filter based on nominal values of plant dynamics may not be robust against modeling uncertainty and disturbances, and sometimes leads to poor performance. Therefore, the research subject of robust estimation and H_∞ estimation has drawn much attention in the past decade, and numerous results have been reported in the literature; see, e.g., [2]–[4], [8], [11], [12], [14], [20], and [21]. However, it is quite common in state estimation problems to have performance objectives that are *directly* expressed as upper bounds on the variances of the estimation error. Several *indirect* approaches attempt to achieve these constraints, for instance the theory of weighted least-squares estimation [15] minimizes a weighted scalar sum of the error variances of the state estimation, but minimizing a scalar sum does not ensure that the multiple variance requirements will be satisfied.

In recent years, the error covariance assignment (ECA) theory [10], [23], [24] was developed to provide an alternative and more straightforward methodology for designing filter gains which satisfy the above performance objectives. This methodology could provide a closed form solution for *directly* assigning the specified steady-state estimation error covariance. Unfortunately, when there is parameter perturbation in plant modelings, no robust behavior on variance-constrained performance along with stability of filtering process can be guaranteed by the ECA theory. Very recently, [17] studied the robust state estimation problem for perturbed discrete-time systems

Manuscript received December 10, 1996. Recommended by Associate Editor, E. Yaz. This work was supported by the Alexander von Humboldt Foundation of Germany and the National Natural Science Foundation of China.

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Publisher Item Identifier S 0018-9286(99)02099-1.

with prespecified H_∞ norm and error variance upper-bound constraints, and it was shown in [17] the construction of the state estimator called for the solution of some complicated nonlinear matrix inequalities, and therefore the design of expected discrete-time estimators was numerically difficult. In the event of state estimation for an inherently time-continuous system in terms of a discrete-time "equivalent," the question of sampling is not trivial [16], since the very small sampling period which is naturally required will result in computational difficulties. Moreover, the parameters in the discrete-time model usually do not correspond to the physical meanings. Thus, the purpose of this paper is to investigate the robust variance-constrained H_2/H_∞ state estimation problem for the uncertain continuous-time systems. It will be seen that the existence conditions as well as the analytical expression of the desired continuous-time estimators are derived in terms of bilinear matrix inequalities (BMI's) which are not difficult to deal with.

It should be pointed out the problem addressed in this paper is very different from that in [3], [4], [12], [20], and [21]. In these papers, an *optimal* robust filter is designed to minimize the upper bound on the variance of the estimation error for all admissible parameter uncertainties, and therefore the resulting optimal filter is usually unique and it seems that there is little freedom to be used to achieve other performances. However, since the specified variance constraints may not be minimal but should meet engineering requirements, the addressed problem in this paper is actually a multiobjective design task which often yields nonunique solutions. After assigning to the system a specified variance upper bound, there exists *much* freedom which can be used to attempt to *directly* achieve other desired performance requirements, such as robustness, H_∞ requirement, transient property, etc., but the traditional optimal (robust) Kalman filtering methods may lack such an advantage.

This paper studies the state estimator design problem for perturbed linear continuous-time systems with H_∞ norm and variance constraints. The perturbation is assumed to be time-invariant and norm-bounded and enters into both the state and measurement matrices. The problem we address is to design a linear state estimator such that, for all admissible measurable perturbations, the variance of the estimation error of each state is not more than the individual prespecified value, and the transfer function from disturbances to error state outputs satisfies the prespecified H_∞ norm upper bound constraint, simultaneously. Existence conditions of the desired estimators are derived in terms of Riccati-type matrix inequalities, and the analytical expression of these estimators is also presented. A numerical example is provided to show the directness and effectiveness of the proposed design approach.

II. Problem Formulation and Assumptions

Consider the following class of linear uncertain observable systems [2]:

$$\dot{x}(t) = (A + \Delta A)x(t) + D_1 w(t) \tag{1}$$

and the measurement equation

$$y(t) = (C + \Delta C)x(t) + D_2 w(t) \tag{2}$$

where x is an n-dimensional state vector, y is an p-dimensional measured output vector, and A, C, D_1, D_2 are known constant matrices. $w(t)$ is a zero mean Gaussian white noise process with covariance $I > 0$. The initial state $x(0)$ has the mean $\bar{x}(0)$ and covariance $P(0)$ and is uncorrelated with $w(t)$. ΔA and ΔC are perturbation matrices which represent parametric uncertainties and assumed to be of the time-invariant form

$$\begin{bmatrix} \Delta A \\ \Delta C \end{bmatrix} = \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} FN \tag{3}$$

where $F \in R^{i \times j}$ is a perturbation matrix which satisfies

$$FF^T \le I \tag{4}$$

and M_1, M_2 (M_2 is full row rank) and N are known constant matrices of appropriate dimensions. ΔA and ΔC are said to be admissible if both (3) and (4) hold.

When the perturbations ΔA and ΔC are measurable, the state estimation vector $\hat{x}(t)$ satisfies the following linear full-order filter:

$$\dot{\hat{x}}(t) = (A + \Delta A)\hat{x}(t) + K[y(t) - (C + \Delta C)\hat{x}(t)] \tag{5}$$

whose estimation error covariance in the steady state is defined as

$$P := \lim_{t \to \infty} P(t) := \lim_{t \to \infty} E[e(t)e^T t], \qquad e(t) = x(t) - \hat{x}(t). \tag{6}$$

Then, from system (1), (2), estimator (5), and definition (6), we have

$$\dot{e}(t) = [A + \Delta A - K(C + \Delta C)]e(t) + (D_1 - KD_2)w(t) \tag{7}$$

and

$$\begin{aligned} \dot{P}(t) = {} & [A + \Delta A - K(C + \Delta C)]P(t) + P(t)[A + \Delta A \\ & - K(C + \Delta C)]^T + (D_1 - KD_2)(D_1 - KD_2)^T. \end{aligned} \tag{8}$$

Define filtering matrix $A_f := A + \Delta A - K(C + \Delta C)$. If A_f is Hurwitz stable (i.e., the poles of A_f all have negative real parts) for all admissible ΔA and ΔC, then in the steady state, the estimation error covariance $P\,(P = P^T > 0)$ satisfies

$$A_f P + P A_f^T + (D_1 - KD_2)(D_1 - KD_2)^T = 0. \tag{9}$$

Our objective in this paper is to deal with the robust filtering problem, i.e., design a filter gain K such that, for all admissible measurable perturbations ΔA and ΔC, the following three requirements are simultaneously satisfied.

1) The filtering matrix $A_f = A + \Delta A - K(C + \Delta C)$ remains Hurwitz stable.
2) The steady-state error covariance P meets
$$[P]_{ii} \le \sigma_i^2, \qquad i = 1, 2, \cdots, n \tag{10}$$
where $[P]_{ii}$ means the ith diagonal element of P, i.e., the steady-state variance of ith state. σ_i^2 $(i = 1, 2, \cdots, n)$ denotes the prespecified steady-state error estimation variance constraint on ith state and can be determined by the practical performance requirements.
3) The H_∞ norm of the transfer function $H(s) = L(sI_n - A_f)^{-1}(D_1 - KD_2)$ from disturbances $w(t)$ to error state outputs $Le(t)$ satisfies the constraint
$$\|H(s)\|_\infty \le \gamma \tag{11}$$
where L is the known error state output matrix and
$$\|H(s)\|_\infty = \sup_{\omega \in R} \sigma_{\max}[H(j\omega)] \tag{12}$$
and $\sigma_{\max}[\cdot]$ denotes the largest singular value of $[\cdot]$; γ is a given positive constant.

III. Main Results and Proofs

Theorem 3.1: Define $A_c := A - KC, \Delta A_c = \Delta A - K(\Delta C)$. Let $\delta > 0$ be arbitrarily small. If there exist a positive definite matrix $Q \in R^{n \times n}$ and a positive scalar $\varepsilon > 0$ satisfying the algebraic matrix equation

$$\begin{aligned} & A_c Q + Q A_c^T + \varepsilon(M_1 - KM_2)(M_1 - KM_2)^T \\ & + Q(\varepsilon^{-1} N^T N + \gamma^{-2} L^T L)Q \\ & + (D_1 - KD_2)(D_1 - KD_2)^T + \delta I = 0 \end{aligned} \tag{13}$$

then for all admissible measurable ΔA and ΔC, we have the following conclusions:

1) the filtering matrix $A_f = A + \Delta A - K(C + \Delta C)$ is robustly stable;

2) the steady-state error covariance P exists and meets $P \leq Q$;
3) $\|H(s)\|_\infty \leq \gamma$ where $H(s) = L(sI_n - A_f)^{-1}(D_1 - KD_2)$.

Proof:

1) By the definitions of A_f, A_c and ΔA_c, it is clear that

$$A_f = A - KC + (M_1 - KM_2)FN = A_c + \Delta A_c. \quad (14)$$

Note that

$$\begin{aligned}&[\varepsilon^{1/2}(M_1 - KM_2)F - \varepsilon^{-1/2}QN^T] \\ &\cdot [\varepsilon^{1/2}(M_1 - KM_2)F - \varepsilon^{-1/2}QN^T]^T \\ &= \varepsilon(M_1 - KM_2)FF^T(M_1 - KM_2)^T + \varepsilon^{-1}QN^TNQ \\ &- [(M_1 - KM_2)FN]Q - Q[(M_1 - KM_2)FN]^T \geq 0\end{aligned}$$

and by the fact that $FF^T \leq I$, the following inequality holds:

$$\begin{aligned}&[(M_1 - KM_2)FN]Q + Q[(M_1 - KM_2)FN]^T \\ &\leq \varepsilon(M_1 - KM_2)(M_1 - KM_2)^T + \varepsilon^{-1}QN^TNQ. \quad (15)\end{aligned}$$

Define

$$\begin{aligned}\Sigma :&= \varepsilon(M_1 - KM_2)(M_1 - KM_2)^T + \varepsilon^{-1}QN^TNQ \\ &- [(M_1 - KM_2)FN]Q - Q[(M_1 - KM_2)FN]^T.\end{aligned}$$

It follows from (15) that $\Sigma \geq 0$. Using the definitions of Σ, A_f, A_c and the relation (14), (13) can be rewritten as

$$\begin{aligned}&A_fQ + QA_f^T + \Sigma + \gamma^{-2}QL^TLQ \\ &+ (D_1 - KD_2)(D_1 - KD_2)^T + \delta I = 0. \quad (16)\end{aligned}$$

Since $\Sigma + \gamma^{-2}QL^TLQ + (D_1 - KD_2)(D_1 - KD_2)^T + \delta I > 0$, it follows from Lyapunov stability theory that the filtering matrix $A_f = A + \Delta A - K(C + \Delta C)$ remains asymptotically stable for all admissible ΔA and ΔC.

2) Since A_f is asymptotically stable, the steady-state error covariance P exists and meets (9). Subtract (9) from (16) to obtain

$$A_f(Q - P) + (Q - P)A_f^T + \Sigma + \gamma^{-2}QL^TLQ + \delta I = 0 \quad (17)$$

or equivalently $Q - P = \int_0^\infty e^{A_f t}(\Sigma + \gamma^{-2}QL^TLQ + \delta I)e^{A_f^T t}dt \geq 0$ which means that $P \leq Q$.

3) Now, we can also rearrange (13), or (16), as the following:

$$\begin{aligned}&A_fQ + QA_f^T + \gamma^{-2}QL^TLQ \\ &+ (D_1 - KD_2)(D_1 - KD_2)^T + \Theta = 0 \quad (18)\end{aligned}$$

where $\Theta := \Sigma + \delta I$. Then, the proof of $\|H(s)\|_\infty \leq \gamma$ can be completed by a standard manipulation of (18); for details see Willems [18, Lemma 1]. This proves Theorem 3.1.

Remark 3.1: The parameter $\delta > 0$ which can be arbitrarily small is only to guarantee that conclusion 1) holds. In the case when $D_1 - KD_2$ is full row rank or the matrix L is nonsingular δ can be set to be zero.

Remark 3.2: Theorem 3.1 implies that the satisfaction of (13) leads to the upper bounds on both the error covariance and H_∞ performance which may be conservative primarily due to the introduction of the additional matrix $\Sigma > 0$. Since $\Sigma > 0$ is the function of parameter $\varepsilon > 0$, an approach to reducing the conservative upper bounds is to appropriately choose $\varepsilon > 0$ [20], [21], and the detailed analysis and algorithm about the minimization of $\|\Sigma\|$ over the scaling parameter $\varepsilon > 0$ can be found in [19].

Remark 3.3: Theorem 3.1 shows that the robust stability and H_∞ constraints on the filtering process are automatically enforced when a positive definite solution to (13) is known to exist. Furthermore, all such solutions provide upper bounds for the actual steady-state estimation error covariance P. This behavior will be utilized to achieve the prescribed upper bounds on estimation error variance.

We now assign a desired value to the positive definite matrix Q such that

$$[Q]_{ii} \leq \sigma_i^2, \qquad i = 1, 2, \cdots, n \quad (19)$$

and then find the set of Kalman filter gain K which satisfies (13) for the specified triple (Q, ε, δ). If such a gain exists and can be parameterized, then for admissible perturbations, it follows from Theorem 3.1 that: (1) A_f is robustly stable; (2) $\|H(s)\|_\infty \leq \gamma$; and 3) $[P]_{ii} \leq [Q]_{ii} \leq \sigma_i^2, i = 1, 2, \cdots, n$. Hence, the variance-constrained robust H_2/H_∞ filtering gain design task will be accomplished, and the problem addressed in Section II can be interpreted as *an auxiliary (Q, ε, δ)—triple achievement" problem*.

Definition 1: Let $\varepsilon > 0$ and $\delta > 0$ be positive scalars where $\delta > 0$ is arbitrarily small, and $Q \in R^{n \times n}$ be a positive definite matrix meeting (19). The specified triple (Q, ε, δ) is said to be achievable if there exists a set of filtering gain K such that the (13) holds for this triple.

Remark 3.4: It is clear from the above analysis that the achievability of a given triple (Q, ε, δ) will result in the finish of the addressed variance-constrained robust H_2/H_∞ filtering gain design task. Hence, in what follows, our purpose is to derive the necessary and sufficient conditions for the existence of an achievable triple (Q, ε, δ) and then to characterize all filtering gains associated with the achievable triple (Q, ε, δ).

Theorem 3.2: Let the desired steady-state error variance constraints $\sigma_i^2 (i = 1, 2, \cdots, n)$ and the H_∞ disturbance attenuation constraint γ be given. Assume that the matrix $Q > 0$ meets (19), and $\varepsilon > 0, \delta > 0$ are positive scalars where $\delta > 0$ is arbitrarily small. Then a triple (Q, ε, δ) is achievable if and only if this triple satisfies the following algebraic Riccati-type inequality:

$$\begin{aligned}&\big[A - \big(\varepsilon M_2M_1^T + D_2D_1^T\big)^T\big(\varepsilon M_2M_2^T + D_2D_2^T\big)^{-1}C\big]Q \\ &+ Q\big[A - \big(\varepsilon M_2M_1^T + D_2D_1^T\big)^T\big(\varepsilon M_2M_2^T + D_2D_2^T\big)^{-1}C\big]^T \\ &+ Q\big[\varepsilon^{-1}N^TN + \gamma^{-2}L^TL - C^T\big(\varepsilon M_2M_2^T + D_2D_2^T\big)^{-1}C\big]Q \\ &+ \big[\varepsilon M_1M_1^T + D_1D_1^T + \delta I - \big(\varepsilon M_2M_1^T + D_2D_1^T\big)^T \\ &\cdot \big(\varepsilon M_2M_2^T + D_2D_2^T\big)^{-1}\big(\varepsilon M_2M_1^T + D_2D_1^T\big)\big] \leq 0 \quad (20)\end{aligned}$$

and the left-hand side of (20) is of maximum rank p.

Proof: For the purpose of simplicity, we first make the following definitions:

$$X := \varepsilon M_2M_2^T + D_2D_2^T \quad (21)$$

$$Y := CQ + \varepsilon M_2M_1^T + D_2D_1^T \quad (22)$$

$$\begin{aligned}Z :&= AQ + QA^T + Q(\varepsilon^{-1}N^TN + \gamma^{-2}L^TL)Q \\ &+ \varepsilon M_1M_1^T + D_1D_1^T + \delta I \quad (23)\end{aligned}$$

and notice that $A_c = A - KC$, (13) can be rewritten in the following simple form:

$$KXK^T - KY - Y^TK^T + Z = 0. \quad (24)$$

Since M_2 is full row rank, then the matrix X is positive definite, and (13), or (24), can be equivalently expressed as follows:

$$\begin{aligned}&-(-KX^{1/2} + Y^TX^{-1/2})(-KX^{1/2} + Y^TX^{-1/2})^T \\ &= Z - Y^TX^{-1}Y. \quad (25)\end{aligned}$$

Since the dimension of filter gain K is $n \times p$ and $p \leq n$, then from (25), there exists a solution K to (13) (i.e., the triple (Q, ε, δ) is achievable) if and only if the right-hand side of (25) is negative semidefinite, i.e.,

$$Z - Y^T X^{-1} Y \leq 0 \tag{26}$$

and is of maximum rank p (in this case, both sides of (25) have compatible ranks). Note that (26) can also be rearranged as the Riccati-type matrix inequality (20), and thus the proof of Theorem 3.2 is completed.

Theorem 3.2 gives the algebraic parameterization for the existence conditions of an achievable triple (Q, ε, δ) in term of a Riccati-type matrix inequality. Furthermore, the algebraic expression of all filtering gains K related to the achievable triple (Q, ε, δ) will be stated in the following theorem.

Theorem 3.3: Suppose that the prespecified triple (Q, ε, δ) is achievable, where the positive definite matrix $Q > 0$ meets (19) and the positive scalar $\delta > 0$ is arbitrarily small. The desired filtering gains which achieve the triple (Q, ε, δ) can be characterized as follows:

$$K = Y^T X^{-1} - TVX^{-1/2} \tag{27}$$

where $T \in R^{n \times p}$ is the square root of $Y^T X^{-1} Y - Z$ (i.e., $TT^T = Y^T X^{-1} Y - Z$), $V \in R^{p \times p}$ is an arbitrary orthogonal matrix, and X, Y, Z are defined in (21), (22), and (23), respectively.

Proof: It follows from (25) and the definition of T that

$$\begin{aligned} -Z + Y^T X^{-1} Y = TT^T &= (-KX^{1/2} + Y^T X^{-1/2}) \\ &\quad \cdot (-KX^{1/2} + Y^T X^{-1/2})^T \end{aligned} \tag{28}$$

or equivalently $TV = -KX^{1/2} + Y^T X^{-1/2}$, and then (27) follows immediately.

Finally, the following result which gives the solution to the variance-constrained robust H_2/H_∞ filtering gain design problem addressed in this paper is easily accessible.

Corollary 3.1: Let the steady-state error variance constraints $\sigma_i^2 (i = 1, 2, \cdots, n)$, the H_∞ disturbance attenuation upper bound constraint γ, and the perturbation structure (3) be prespecified. If a triple (Q, ε, δ) is achievable where matrix $Q > 0$ meets (19) and scalar $\delta > 0$ is arbitrarily small, i.e., the conditions of Theorem 3.2 are satisfied, then the desired variance-constrained robust H_2/H_∞ filtering gains can be obtained by (27).

Remark 3.5: We can see that the necessary and sufficient conditions for the achievability of a given triple (Q, ε, δ) are very easy to test. On the other hand, in practical applications the designers often wish to construct the achievable triple (Q, ε, δ) directly from the achievability conditions and then get the desired filtering gains from (27) rapidly. Noting that the condition (20) on achievable triple (Q, ε, δ) is actually a BMI of parameter Q, we can first choose appropriate $\varepsilon > 0$ which reduce (minimize) the possible conservative upper bounds on both the error covariance and H_∞ performance in Theorem 3.1 by applying the method proposed in [19] and [20] and choose sufficiently small positive scalar $\delta > 0$ in certain cases, and then solve BMI (20) for parameter Q by using the algorithms guaranteeing global convergence [6], [22], as well as local numerical searching algorithms that converge (without a guarantee) much faster [5], [7]. Also, some properties of BMI's can be found in [13].

Remark 3.6: It is clear that if the set of desired estimators is not empty, it must be very large. We may utilize the freedom in estimator design to improve other system properties. An interesting problem for future research is how to exploit the freedom to achieve the specified transient constraint and reliable constraint on filtering process.

IV. Numerical Example

Consider a perturbed linear continuous-time stochastic system described by (1)–(3), where the parameters are assumed to be the following:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad L = \begin{bmatrix} 0.5 & 0.1 \\ 0 & 1.6 \end{bmatrix}$$

$$D_1 = \begin{bmatrix} 0.0563 & 0 \\ 0 & 0.0792 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 0.0911 & 0 \\ 0 & 0.1572 \end{bmatrix}$$

$$\Delta A = M_1 F N = \begin{bmatrix} 0.0126 & 0.0457 \\ 0.0068 & 0.4369 \end{bmatrix} \begin{bmatrix} \sin\alpha & 0 \\ 0 & \cos\alpha \end{bmatrix} \cdot \begin{bmatrix} 0.3467 & 0.0546 \\ 0.0005 & 0.0121 \end{bmatrix}$$

$$\Delta C = M_2 F N = \begin{bmatrix} 0.0012 & 0.0001 \\ 0.0034 & 0.0064 \end{bmatrix} \begin{bmatrix} \sin\alpha & 0 \\ 0 & \cos\alpha \end{bmatrix} \cdot \begin{bmatrix} 0.3467 & 0.0546 \\ 0.0005 & 0.0121 \end{bmatrix}.$$

We aim at designing robust filtering gains such that 1) the filtering matrix $A_f = A - KC + \Delta A - K(\Delta C)$ is robustly stable; 2) the steady-state covariance P exists and $[P]_{11} \leq \sigma_1^2 = 0.04, [P]_{22} \leq \sigma_2^2 = 0.2$; and 3) the transfer function $H(s)$ from disturbances $w(t)$ to error state outputs $Le(t)$ satisfies the constraint $\|H(s)\|_\infty \leq \gamma = 0.8259$.

Now, based on Remark 3.2, we first exploit the approach proposed in [19] and [20] to reduce the conservativeness of present results, and hence an appropriate scalar $\varepsilon > 0$ can be found as $\varepsilon = 0.5017$. Then, based on Remark 3.1, since the error state output matrix L is nonsingular, the constant δ may be set as $\delta = 0$. Moreover, using the algorithm discussed in Remark 3.5, we can solve inequality (20) and obtain

$$Q = \begin{bmatrix} 0.0271 & 0.0543 \\ 0.0543 & 0.1406 \end{bmatrix}, \quad T = \begin{bmatrix} 0.3468 & 0.0002 \\ 1.0673 & 0.0322 \end{bmatrix}$$

and therefore

$$X = \begin{bmatrix} 0.0083 & 0.0000 \\ 0.0000 & 0.0247 \end{bmatrix}, \quad Y = \begin{bmatrix} 0.0322 & 0.0544 \\ 0.0544 & 0.1545 \end{bmatrix}$$

$$Z = \begin{bmatrix} 0.1249 & 0.1811 \\ 0.1811 & 0.1802 \end{bmatrix}.$$

Obviously, $[Q]_{11} \leq \sigma_1^2 = 0.04, [Q]_{22} \leq \sigma_2^2 = 0.2$, i.e., the condition (19) is satisfied. Then, substituting the triple (Q, ε, δ) and the orthogonal matrices $V_1 = I_2, V_2 = \mathrm{diag}(1, -1), V_3 = \mathrm{diag}(-1, 1), V_4 = -I_2$ into the expression (27), respectively, lead to the following desired filtering gains:

$$K_1 = \begin{bmatrix} 0.0752 & 2.2018 \\ -5.1690 & 6.0403 \end{bmatrix}, \quad K_2 = \begin{bmatrix} 0.0752 & 2.2043 \\ -5.1674 & 6.4488 \end{bmatrix}$$

$$K_3 = \begin{bmatrix} 7.6885 & 2.2013 \\ 18.2630 & 6.0379 \end{bmatrix}, \quad K_4 = \begin{bmatrix} 7.6885 & 2.2039 \\ 18.2630 & 6.4474 \end{bmatrix}$$

and it is not difficult to test that the prespecified robust stability and H_∞ norm constraints on filtering process and the variance constraint on estimation error are all met.

V. Conclusion

In this paper we have considered a robust variance-constrained H_2/H_∞ state estimation problem for linear continuous-time systems with parameter perturbations in both the state and measurement matrices. It has been shown that this robust estimation problem can be converted to an auxiliary "triple (Q, ε, δ) achievement" problem which is related to solutions of a Riccati-type BMI. Based on this BMI, the existence conditions and the analytical expression of desired estimators have been characterized. These results are analogous to results obtained previously for the perturbed discrete-time systems.

We finally remark that the extension of the results developed in the present paper to the variance-constrained multiobjective (e.g., robustness, transient behavior, H_∞ requirement, fault-tolerant property) state estimation for various systems such as continuous-time, discrete-time, sampled-data, and stochastic parameter systems still remains to be investigated within the framework of this approach.

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The Relationship Between Minimum Entropy Control and Risk-Sensitive Control for Time-Varying Systems

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***Abstract*— In this paper, the connection between minimum entropy control and risk-sensitive control for linear time-varying systems is investigated. For time-invariant systems, the entropy functional and the linear exponential quadratic Gaussian cost are the same. In this paper, it is shown that this is not true for general time-varying systems. It does hold, however, when the system admits a state-space representation.**

***Index Terms*— Discrete-time systems, entropy, risk-sensitive control, time-varying systems.**

I. INTRODUCTION

Controllers which minimize the entropy of the closed-loop transfer function have been studied extensively for linear time-invariant (LTI) systems, both in the continuous and discrete-time cases [1]–[3]. It has been shown that these controllers have considerable advantages over the standard optimal $\mathcal{H}_\infty$ controllers. Among these advantages, the minimum entropy requirement imposes uniqueness in the controllers which achieve an induced norm bound. Second, there is considerable interest in finding a controller which minimizes the closed-loop $\mathcal{H}_2$ norm, while ensuring an $\mathcal{H}_\infty$ bound on the system. To date, this remains an open problem. Nevertheless, it can be shown that the entropy is an upper bound for the $\mathcal{H}_2$ norm of the system, and thus minimum entropy controllers provide a degree of $\mathcal{H}_2$ norm performance that is sacrificed in other $\mathcal{H}_\infty$ controllers.

In a series of recent papers, a theory of minimum entropy control for time-varying systems has been formulated [4], [5]. A straightforward extension is nontrivial due to the fact that the entropy is defined in terms of the closed-loop system's transfer function, which other types of systems do not admit. However, for discrete-time systems, an operator theoretic formulation of the problem, together with some fundamental factorizations theorems on nest algebras, have allowed us to expand the definition of entropy.

Unlike the time-invariant case, where the entropy is defined as a scalar number, the entropy for a time-varying system is defined as a memoryless operator. This is intuitively appealing due to the time-varying nature of the system. Nevertheless, this notion of entropy has the unsatisfying property that the entropy of a causal system does not equal the entropy of its anticausal adjoint. This has the effect of requiring a considerably different approach from that used for LTI systems in the solution of the minimum entropy control problem [5].

In this paper, we address the relationship between the minimum entropy control problem for time-varying systems and the stochastic

Manuscript received May 26, 1998. Recommended by Associate Editor, T. Runolfsson. This work was supported in part by the National Science Foundation under Grant ECS-9309387.

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Publisher Item Identifier S 0018-9286(99)03025-1.