Systems Modelling and Simulation (4)

Markov Process and its relationship with Discrete Event Simulation

Markov Process

- One of the most important subjects in Discrete Event Simulation *(Queuing principles)*
- Markov processes are powerful tools for describing dynamic systems that are stochastic in nature
- Markov processes constitute the fundamental theory in describing queues

Markov Simply

- Describes an event that depends on the immediate event before it.
- Example tossing a coin which one event is totally independent of the one before it, as opposed to a Card Game such as Trumps that the card played is highly dependent on the card played before

Markov Process continued

A stochastic process $X_t : t \in T$ and $T \subseteq N_+ = [0, \infty]$

Is a Markov process if for all $t_0 = 0,\ t_0 < t_1 < t_2 < \ldots < t_n < t_{n+1}$

and all $s \in S$ the cumulative density function of $X_{t_{n+1}}$

is dependent on the last value of X_{t_n} and **not** on earlier values.

A Markov process is therefore a conditional probability:

$$P(X_{t_{n+1}} \leq s_{n+1} \mid X_{t_n} = s_n , X_{t_{n-1}} = s_{n-1} ,\ldots, X_{t_0} = s_0)$$
$$= P(X_{t_{n+1}} , s_{n+1} \mid X_{t_n} , s_n)$$

Markov Chains

- Markov Processes can be:
  1. **Homogenous (time independent)**
  2. **Non-homogenous (time dependent)**

- Parameters of Markov process:
  1. **Discrete**
  2. **Continuous**

- Markov process with discrete state spaces are called **Markov Chains**

Discrete Time Markov Chains

A stochastic process of $(X_0, X_1, X_2, \ldots, X_n, X_{n+1}, \ldots)$

at points of observation *0,1,2,…, n, n+1, …*

Constitute a Discrete Time Markov Chain provided that the conditional probability mass function (*pmf*) is as:

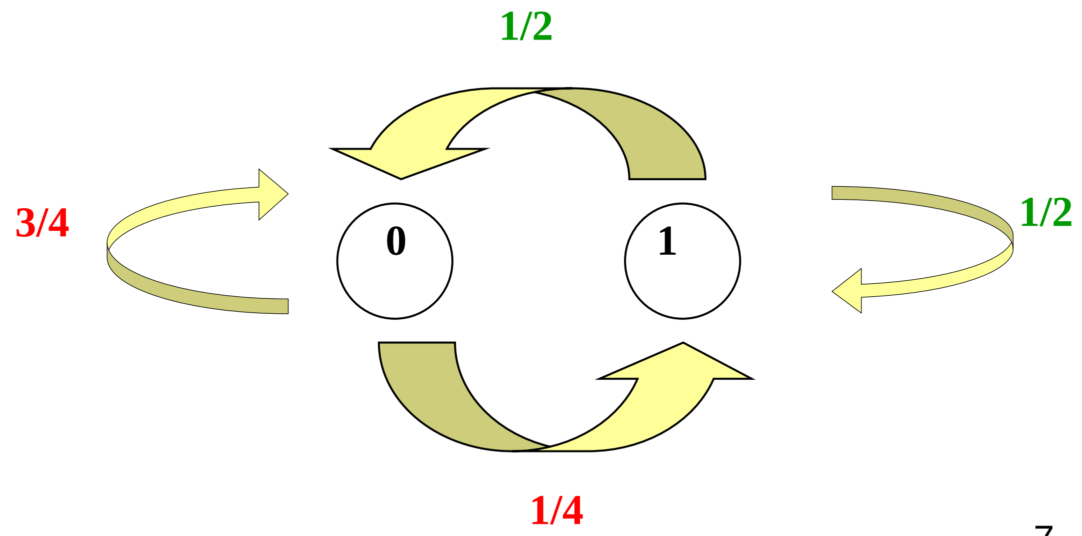
$$P(X_{n+1} = s_{n+1} \mid X_n = s_n, X_{n-1} = s_{n-1}, \ldots, X_0 = s_0)$$
$$= P(X_{n+1} = s_{n+1} \mid X_n) = s_n$$

Therefore a Markov Chain

- Evolution from state s_0 to s_n is step by step with a transition probability

Diagram explains a Discrete Time Markov Chain

$$P^1 = \begin{bmatrix} 0.75 & 0.25 \\ 0.5 & 0.5 \end{bmatrix}$$



Example

The probabilities of weather conditions, given the weather on the preceding day, can be represented by the state transition matrix:

$$P^1 = \begin{bmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{bmatrix} = \begin{bmatrix} 0.9 & 0.1 \\ 0.5 & 0.5 \end{bmatrix}$$

State 0 = sunny and State 1 = Rainy.

Reading the matrix, the probability of a day being sunny and the following to be sunny is 0.9. The probability of sunny to rainy will be the remaining 0.1. Can you decipher the second row?

Example continued

If on day 0 the weather is sunny, then meaning the day is sunny then 100% and rainy 0%.

To predict weather in day 1:

$$X^{(1)} = X^{(0)}.\mathrm{P} \Rightarrow (1 \quad 0)\begin{bmatrix} 0.9 & 0.1 \\ 0.5 & 0.5 \end{bmatrix} = (1 \times 0.9 + 0 \times 0.5 \quad 1 \times 0.1 + 0 \times 0.5) = (0.9 \quad 0.1)$$

Day 2

$$X^{(2)} = X^{(1)}\mathrm{P} = X^{(0)}\mathrm{P}^{2} = (0.9 \quad 0.1)\begin{bmatrix} 0.9 & 0.1 \\ 0.5 & 0.5 \end{bmatrix} = (.9 \times .9 + .1 \times .5 \quad .9 \times .1 + .1 \times .5) = (.86 \quad .14)$$

Example continued

In long run this terrain when number of *n* goes to infinity the steady state will be

$$(\,p_0 \quad p_1\,) = (\,0.833 \quad 0.167\,)$$

in other words if you want to bet on a day to be sunny or rainy in the future you better put your money on a sunny day!

Do you think bankers and stock exchange people follow similar models?! Probably <u>real clever</u> ones do!!!!

Markovian Queues

- The randomness of our activities and environment
- We all have experienced queues
- Queues have rule (e.g. FCFS, NCFS, LCFS, LVF, …)
- Queues form due to the difference between inter-arrival time and process time
- This extends to all systems: Service, Manufacturing, Supply Chain, Distribution, Logistics, Communication Network, Computing, Traffic and Transport,…

Measuring number of Jobs in a Queue

Three key data is required ***(A,B,m)***:

- ***A*** *is for* distribution function of inter-arrivals
- ***B*** is for distribution function of processing time
- ***m*** is the number of servers

M/M/1 => *random arrivals, random process time, 1 server*

M/M/c => *random arrivals, random process time, c servers*

Server Utilisation Factor

If λ is the average arrival rate and μ is the average processing time for *c* servers the utilisation factor will be:

$$\rho = \frac{\lambda}{\mu c}$$

Assuming that the probability of *n* parts to be at a workstation at time *t to be* $p_t(n)$ in long run *(steady state)* will then be

$$p_t(n) = p_{t+\delta t}(n)$$

Queuing results for an *M/M/1* situation

	Notation	*M/M/1*
Probability of 0 jobs at the workstation	$p(0)$	$1-p$
Expected no. of Jobs waiting in Queue	L_q	$\frac{\rho^2}{1-\rho}$
Expected no. of jobs at workstation	L	$\frac{\rho}{1-\rho}$
Expected Queuing Time	W_q	$\frac{\rho}{\mu(1-\rho)}$
Expected Throughput time	W	$\frac{1}{\mu(1-\rho)}$

Example of an *M/M/1*

A security and metal detection machine at an airport has a service rate that follows an Exponential distribution with. $\mu = 10$ minute

Passengers arrive at the machine with an Exponential rate of $\lambda = 8$ minute. The queuing rule is FCFS. Find the expected machine utilisation, passenger throughput time and average waiting time?

Solution

- Treat as an *M/M/1* system

$$\text{Machine Utilisation:} \quad \rho = \frac{\lambda}{c\mu} = \frac{8}{10}$$

$$\text{Probability that the machine would be idle:} \quad p(0) = 1 - 0.8 = 0.2$$

$$\text{Throughput time:} \quad W = \frac{1}{\mu(1-\rho)} = \frac{1}{10(1-0.8)} = 0.5m$$

$$\text{Waiting time:} \quad W_q = \frac{\rho}{\mu(1-\rho)} = \frac{0.8}{2} = 0.4m$$

Today

- We discussed Markov Processes
- Markov Chains
- Markovian Queues
- How to measure system performance using queuing models

Next week Discrete Simulation Environment