Simple Asset Pricing

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1 Introduction

This page contains a graphical simulation of the stochastic differential equation,

$$\frac{dS}{S} = \mu dt + \sigma dX, \tag{1}$$

where dt is an infinitesimal time increment and dX is a random variable drawn from $N(0, dt)$, a normal distribution with zero mean and standard deviation $\sqrt{dt}$ (dX is a *Wiener process*).

This equation is often used to model stock prices and then the drift, μ, is the expected rate of return on the asset, such as the interest on a risk-free bank deposit (e.g. $\mu = 0.15$ for a 15% interest rate). The volatility, σ, represents the standard deviation on the return dS/S and the random variable dX is included to model stock market uncertainty. See [2] or [5] for details.

*This web page started life in February 1999. Since then Marcus has left Brunel and the URL given here may no longer be active.

It must be emphasized that (1) is only an **intuitive** model. Stock prices behave in a very complicated way and depend on a great many factors: they neither know nor care anything about differential equations and so, as the man said (`www.brunel.ac.uk/~icsrsss/sounds/cannot.wav"`), we have to use our intuition.

2 Using the applet

NOTE: for reasons I have not yet figured out this applet doesn't work properly in Netscape running under GNU/Linux.

To use the java applet follow these steps:

- Enter the initial condition, $S(0) = S_0$, and the stopping time, T. (Press "enter" each time.)
- Enter the drift μ, and the volatility σ. (Press "enter" each time.)
- Select the colour of the graph, the accuracy (No. of time steps) and press "Plot".

Note that to change either S_0 or T you must press "clear" and start again—otherwise the axes will not be correct.

You must press the "enter" key after changing a TextField value, otherwise Java doesn't realise that you have altered anything (i.e. you must generate an *event* on the component).

3 Statistics for the asset price increment

For the stochastic differential equation,

$$dW = Bdt + AdX,$$

where $dX \sim N(0, dt)$, we can calculate expectations (with respect to $N(0, dt)$) using $E(dX) = 0$ and $E(dX^2) = dt$. This gives:

$$\begin{aligned} E(dW) &= E(AdX + Bdt) = AE(dX) + Bdt = Bdt, \\ E(dW^2) &= E(A^2dX^2 + 2ABdXdt + B^2dt^2) = A^2dt + B^2dt^2, \\ \mathrm{Var}(dW) &= E(dW^2) - E(dW)^2 = A^2dt. \end{aligned}$$

Now, what about (1)? We can fit this into these calculations by choosing $A = \sigma S$ and $B = \mu S$, thus:

$$\begin{aligned} E(dS) &= \mu S dt, \\ E(dS^2) &= \sigma^2 S^2 dt + \mu^2 S^2 dt^2, \\ \mathrm{Var}(dS) &= \sigma^2 S^2 dt. \end{aligned}$$

Convinced? you shouldn't be! These are in fact correct but it is far from obvious why they are. In our first set of calculations we tacitly assumed that A and B were *deterministic* (i.e. non-random), and this allowed us to take them "outside" of the expectaion operator. In the second set of calculations A and B both depend on S, a stochastic process (and therefore random), so how can we justify taking A and B (i.e. S) outside of the expectation operator?

The answer to this riddle lies with the manner in which we interpret (1). The key is that dS is the increment *away* from $S(t)$ during the small time dt. Thus, when we take the expectations $S(t)$ is *already known*, and, therefore, non-random.

Strictly speaking, the expectaions above are *conditional* on the information $S(t)$ being known at time t.

4 The lognormal distribution

Denote by $N(\mu, \sigma^2)$ the normal distribution with mean μ and variance σ^2. If X is a random variable with such a normally distributed logarithm: $\ln X \sim N(\mu, \sigma^2)$, then X is said to be *lognormally* distributed.

To determine the probability density function, $q(x)$, for the lognormal distribution we define $Y := \ln X$, then $Y \sim N(\mu, \sigma^2)$, and the probability measure for $N(\mu, \sigma^2)$ is,

$$p(y)dy = \frac{1}{\sigma\sqrt{(2\pi)}} \exp\left(-\frac{(y-\mu)^2}{2\sigma^2}\right) dy \qquad \text{for } y \in (-\infty, \infty).$$

Now, using $x = e^y$, so that $dy = dx/x$, we can find the probability measure for X by the change-of-measure formula,

$$p(y)dy = p(\ln x)\frac{dx}{x} = q(x)dx.$$

Filling in the details we therefore have,

$$\begin{aligned} p(y)dy &= \frac{1}{\sigma\sqrt{(2\pi)}} \exp\left(-\frac{(y-\mu)^2}{2\sigma^2}\right) dy, \\ &= \frac{1}{x\sigma\sqrt{(2\pi)}} \exp\left(-\frac{(\ln x-\mu)^2}{2\sigma^2}\right) dx, \\ &= q(x)dx. \end{aligned}$$

Therefore, if $\ln X \sim N(\mu, \sigma^2)$, then X itself has the probability density function,

$$q(x) = \frac{1}{x\sigma\sqrt{(2\pi)}} \exp\left(-\frac{(\ln x-\mu)^2}{2\sigma^2}\right), \qquad \text{for } x \in (0, \infty).$$

This is the density function for the lognormal distribution, and we denote it by $e^N(\mu, \sigma^2)$.

After some integrations one can show that the moment generating function for this distribution is,

$$M(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \exp\left(\frac{n}{2}(2\mu + n\sigma^2)\right).$$

The centered moments now follow by differentiation "at $t = 0$", and give the following expectations with respect to $e^N(\mu, \sigma^2)$:

$$\begin{aligned} E(X) &= e^{\mu+\sigma^2/2}, & (2) \\ E(X^2) &= e^{2\mu+2\sigma^2}, & (3) \\ &\vdots \\ E(X^m) &= e^{\mu m+m^2\sigma^2/2}. & (4) \end{aligned}$$

The variance is then,

$$\mathrm{Var}(X) = E(X^2) - E(X)^2 = e^{2\mu+\sigma^2}\left(e^{\sigma^2} - 1\right). \tag{5}$$

These calculations are useful when choosing parameters in the binomial tree option pricing technique (see, for example, `www.brunel.ac.uk/~icsrsss/finance/options/binomial`). The reason for this is demonstrated in Section 6, but to make sense of that section we first need *Ito's lemma.*

5 Ito's lemma

Equation (1) is one of the few *Stochastic Differential Equations* to have a closed-form solution. It is:

$$S(t) = \exp\left((\mu - \frac{1}{2}\sigma^2)t + \sigma X(t)\right), \tag{6}$$

and this can be confirmed with Ito's lemma.

Ito's lemma is often referred to as *Taylor's series for stochastic calculus.* This is not quite right, unless one interprets the "differences" in Taylor's series as a kind of shorthand for definite integrals. Stochastic differential equations don't exist (at least not in the way classical differential equations exist), and expressions such as (1) are only a shorthand for Ito integrals. Equation (1) actually means,

$$S(t) = S(\tau) + \int_\tau^t \mu S\,dt + \int_\tau^t \sigma S dX,$$

for $\tau \leq t$ and where the rightmost integral is taken in the sense of Ito. Neftci, [3], is an excellent source for this stuff, but the other popular books, [5, 4, 1] for example, also outline Ito calculus in treatments of various depths.

All we are attempting here is a summary treatment, so here goes. Define a process W by,

$$dW = A(W,t)dt + B(W,t)dX,$$

where $dX \sim N(0, dt)$ (normal: mean zero, variance dt) is a Wiener process and A and B are known, well-behaved, functions. Given a function $f(W,t)$ Ito's lemma gives us the process followed by f and, in essence, is the following result (probably more famous in the stock exchanges than the universities!),

$$df = \left(\frac{\partial f}{\partial t} + A(W,t)\frac{\partial f}{\partial W} + \frac{B(W,t)^2}{2}\frac{\partial^2 f}{\partial W^2}\right)dt + B(W,t)\frac{\partial f}{\partial W}dX. \tag{7}$$

To confirm that (6) is indeed the solution of (1) we use Ito's lemma with $S = f$, $A = 0$ and $B = 1$, giving $W = X$ and,

$$\begin{aligned} dS &= \left(\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial X^2}\right) + \frac{\partial f}{\partial X}dX, \\ &= \left((\mu - \frac{1}{2}\sigma^2)S + \frac{1}{2}\sigma^2 S\right)dt + \sigma S dX, \\ &= \mu S dt + \sigma S dX. \end{aligned}$$

Easy! Well, not really. We didn't actually *solve* the stochastic differential equation, we only verified that we had a solution. In general this is about as good as it gets. You have to get the solution by luck, guesswork, intuition, … [1], and then use Ito's lemma to verify that it really is a solution. Stochastic differential equations are *much harder* to solve than their classical counterparts.

In the next section we look a little harder at the solution of (1) and confirm that S is lognormally distributed.

6 The stochastic differential equation

Starting with (1) let $f(S,t)$ be a function of S and t, then Ito's lemma, in the form,

$$df = \sigma S \frac{\partial f}{\partial S} dX + \left(\mu S \frac{\partial f}{\partial S} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 f}{\partial S^2} + \frac{\partial f}{\partial t} \right) dt,$$

can be used to find the process (i.e. the stochastic differential equation) followed by f.

For example, let $f = \ln S$, then

$$\frac{\partial f}{\partial S} = \frac{1}{S}, \qquad \frac{\partial^2 f}{\partial S^2} = \frac{-1}{S^2} \qquad \text{and} \qquad \frac{\partial f}{\partial t} = 0.$$

Using these in the Ito formula gives,

$$df = \left(\mu - \frac{1}{2}\sigma^2 \right) dt + \sigma dX.$$

Since $dX \sim N(0, dt)$ we can perform calulations similar to those in Section 3 and conclude that,

$$df \sim N((\mu - \sigma^2/2)dt, \sigma^2 dt).$$

Integrating from $\tau < t$ to t and setting $k := t - \tau$ we get,

$$f(t) - f(\tau) \sim N((\mu - \sigma^2/2)k, \sigma^2 k),$$

or, translating the mean,

$$f(t) \sim N(f(\tau) + (\mu - \sigma^2/2)k, \sigma^2 k).$$

Therefore $\ln S(t)$ is normally distributed with

$$\text{mean } = \ln S(\tau) + (\mu - \sigma^2/2)k,$$

(using $f(\tau) = \ln S(\tau)$) and,

$$\text{variance } = \sigma^2 k.$$

In other words, $S(t)$ is *lognormally* distributed.

Some (conditional on information at τ) statistics for $S(t)$ therefore follow from equation (4) once we make the *simultaneous* replacements:

$$\begin{aligned} \mu &\longleftarrow \ln S(\tau) + \left(\mu - \frac{\sigma^2}{2} \right) k, \\ \sigma^2 &\longleftarrow \sigma^2 k. \end{aligned}$$

[1] cheating, crime, old-fashioned hard work, … (you get the picture).

Thus:

$$\begin{aligned} E(S(t)|S(\tau)) &= S(\tau)e^{\mu k}, \\ E(S(t)^2|S(\tau)) &= S(\tau)^2 e^{2\mu k+\sigma^2 k}, \\ \mathrm{Var}(S(t)|S(\tau)) &= S(\tau)^2 e^{2\mu k}(e^{\sigma^2 k} - 1). \end{aligned}$$

These results are useful in constructing a *Binomial Tree* (see e.g. `www.brunel.ac.uk/~icsrsss/finance/options/binomial`) of lognormally distributed asset prices. In this case one would take $t = t_i$, $\tau = t_{i-1}$ and let k be the timestep.

7 PostScript (print quality) source

If I have remembered to generate it, you can obtain a PostScript file of this web page at `www.brunel.ac.uk/~icsrsss/finance/options/sde/home.ps`.

References

[1] Martin Baxter and Andrew Rennie. *Financial calculus.* Cambridge University Press, 1996.

[2] John C. Hull. *Options, futures and other derivative securities.* Prentice-Hall International, Inc., 1993.

[3] Salih N. Neftci. *An introduction to the mathematics of financial derivatives.* Academic Press, 1996.

[4] Paul Wilmott. *Derivatives: the theory and practice of financial engineering.* John Wiley and Sons Ltd, 1998. University Edition.

[5] Paul Wilmott, Sam Howison, and Jeff Dewynne. *The mathematics of financial derivatives.* Cambridge University Press, 1995.