Simple Models of Waiting Lists

G. J. Rodgers[1], Y. J. Yap[1] and T. P. Young[2]

[1]*Department of Mathematical Sciences, Brunel University, Uxbridge, Middlesex UB8 3PH, U.K.*

[2]*Department of Information Systems and Computing, Brunel University, Uxbridge, Middlesex UB8 3PH, U.K.*

Abstract

Motivated by recent empirical studies of the length distribution of hospital waiting lists, we introduce and solve a set of models that imitate the formation of waiting lists. Patients arriving in the system must choose a waiting list to join, based on its length. At the same time patients leave the lists as they get served. The model illustrates how the power-law distributions found in the empirical studies might arise, but indicates that the mechanism causing the power-laws is unlikely to be the preferential behaviour of patients or their physicians.

Keywords: Waiting lists, healthcare, power-law

PACS numbers: 02.50.cw, 05.40.-a, 89.75Hc.

I. INTRODUCTION

Physicists have, for some time, been applying ideas from statistical physics to biological, financial, economic and social systems. In this paper we extend this range of activities to include a study of a healthcare system. In particular we examine a system waiting lists, motivated by the considerable interest in hospital waiting lists in Europe and particularly in the UK. This interest can be seen from the number of software packages available to help healthcare managers plan healthcare provision (see, for instance, [1]).

A number of studies have been carried out on the length distribution of hospital waiting lists, which have shown that this distribution is power-law [2,3]. To support a view that patients or their physicians preferentially select shorter waiting lists, it has also been shown that there is a correlation between the referral rate to lists and the length of the waiting list [4]. This correlation had a time-lag of a length similar to the amount of time it would take information about the length of a list to spread through the system.

Power-law distributions normally signal that a system is self-organising [5], and thus resistant to changes. This implies that, for instance, minor changes in the rate with which patients are treated, or the number of consultants, will have little effect on the distribution of waiting lists, or on the length of the longest list in the system. This question is clearly important to many systems, including those in healthcare; can greater efficiency be addressed by minor changes to the current system, or is a radical overhaul to our whole approach required?

Similarly, if the distribution is power-law, then the average queue length, and the length of the longest queue, are dependent on the value of the exponent in the power-law. It would thus seem to be important to build microscopic models to examine this system and determine under what conditions power-laws occur and the dependence of the power-law exponent on other parameters in the system.

In this paper we attempt to understand the recent empirical results on waiting lists by introducing a toy model of waiting lists which whilest being very simple, we hope captures the essence of the problem. In our model the lists grow or shrink in length as patients join the list to be treated or leave the list after treatment. We find that the model illustrates how power-law distributions might arise, but indicates that the mechanism is unlikely to be the preferential behaviour of patients or their physicians. Although queueing theorists have been interested in problems of this general type for many years [6], they have seldom considered models specifically like ours, in which there are an infinite number of service channels and queue-length dependent rates. Consequently there are very few results in the literature on the distribution of queue lengths in a system with an infinite number of channels.

In the next section we introduce our model and then solve it for a few particular cases. The final section of this paper contains a discussion of our results.

II. THE MODEL

In this section we introduce a model with the minimal ingredients necessary to capture the basic empirical measurements from hospital waiting lists. In particular, we imagine a system in which at time t there are $n_k(t)$ waiting lists of length k. Thus there are

$$\sum_{k=1}^{\infty} k n_k(t) \tag{1}$$

patients waiting in the lists at time t. At each time step a patient in a list of length k is treated with rate A_k and a new patient joins a list of length k with rate B_k. With rate r a new waiting list is created of length zero. The quantity $n_k(t)$ satisfies the equation

$$\frac{dn_k(t)}{dt} = A_{k+1}n_{k+1} - A_k n_k + B_{k-1}n_{k-1} - B_k n_k + r\delta_{k,0} \tag{2}$$

for $k = 0, 1, 2...$ where

$$A_k = \frac{pa_k}{\sum_{k=0}^{\infty} a_k n_k(t)} \tag{3}$$

and

$$B_k = \frac{qb_k}{\sum_{k=0}^{\infty} b_k n_k(t)}. \tag{4}$$

The rate parameters p, q and r allow one to adjust the overall relative rates of patient treatment, arrival and of list creation. The time-independent functions a_k and b_k model, respectively, the rates of leaving and joining a list of length k. The function a_k is the service rate in a list of length k, and b_k controls the propensity of patients to join a list of length k, and models the patients' desire to join shorter lists.

The first term on the right hand side of Eq.(2) is the gain in the number of lists of length k when a patient in a list of length $k+1$ is treated. Similarly, the second term is the loss in the number of lists of length k when a patient in a list of length k is treated. The third term is the gain in the number of lists of length k when a new patient joins a list of length $k-1$. The fourth term is the corresponding loss term. The final term represents the creation of a new list of length 0.

The normalisations in Eqs.(3) and (4) are easy to understand. Consider for example the third term on the right hand side of Eq.(2). A new patient joins a particular list of length $k-1$ with a probability per time step

$$\frac{b_{k-1}}{\sum_{k=0}^{\infty} b_k n_k}. \tag{5}$$

Hence the rate of creation of a list of length k by patient arrival is equal to the rate of patient arrival, q, multiplied by the number of lists of length $k-1$, n_{k-1}, multiplied by the probability of a patient joining a list of length $k-1$, Eq.(5) above. This product gives $B_{k-1}n_{k-1}$, the third term on the right hand side of Eq.(2). Analogous arguments can be made to derive the other terms. Equations similar to Eq.(2) have been used within econophysics [7] and in the field of random growing networks [8].

In the rest of this paper, we will choose p, q, r, a_k and b_k in order to model various scenarios relevant to the modeling of healthcare systems. The functions a_k and b_k can, in principle, take any value, although we must always have $A_0 = a_0 = 0$ because a patient cannot be removed from a waiting list of length 0.

Case 1 $a_k = k$ and $b_k = 1$.

The rates $a_k = k$ and $b_k = 1$ correspond to a situation in which patients join a list at a constant rate, independent of its length. At each time step every patient is equally likely to be removed from the lists, irrespective of the length of the list the patient is in. This choice simplifies Eqs.(2-4) considerably. They then become

$$\frac{dn_k(t)}{dt} = \frac{p}{M(t)}\left[(k+1)n_{k+1} - kn_k\right] + \frac{q}{N(t)}\left[n_{k-1} - n_k\right] + r\delta_{k,0} \tag{6}$$

where

$$N(t) = \sum_{k=0}^{\infty} n_k(t) \qquad \text{and} \qquad M(t) = \sum_{k=0}^{\infty} kn_k(t). \tag{7}$$

It is then a simple matter to show that

$$\frac{dN(t)}{dt} = r \quad \text{and} \quad \frac{dM(t)}{dt} = q - p. \tag{8}$$

In this system, in the large time limit, when $q > p$ and $r > 0$, the $n_k(t), k = 0, 1, 2, \ldots$ and the moments $N(t)$ and $M(t)$ are all linear with time. We write

$$n_k(t) = ts_k \tag{9}$$

and find that s_k obeys

$$s_k = \frac{p}{q-p}\left[(k+1)s_{k+1} - ks_k\right] + \frac{q}{r}\left[s_{k-1} - s_k\right] + r\delta_{k,0}. \tag{10}$$

This equation can be solved using a generating function, $g(\omega)$, defined by

$$g(\omega) = \sum_{k=0}^{\infty} \omega^k s_k \tag{11}$$

which obeys the first order differential equation

$$g(\omega) = (1-\omega)\left[\frac{p}{q-p}\frac{dg}{d\omega} - \frac{q}{r}g(\omega)\right] + r. \tag{12}$$

This differential equation is easily solved to reveal

$$g(\omega) = \frac{r(q-p)}{p}(1-\omega)^{\frac{p-q}{p}} e^{\frac{q\omega}{pr}(q-p)} \int_{\omega}^{1} (1-x)^{\frac{q}{p}-2} e^{\frac{q(p-q)}{pr}x} dx. \tag{13}$$

In the limit $\omega \to 1$ we have

$$g(1) - g(\omega) \sim (q-p)(1-\omega) \tag{14}$$

and hence

$$s_k \sim k^{-2} \qquad \text{as} \qquad k \to \infty. \tag{15}$$

Thus the distribution of waiting lists behaves as $n_k \sim k^{-\beta}$, for large k, with $\beta = 2$ independent of p, q and r. Thus this scenario produces a distribution of waiting lists that is asymptotically power-law.

Case 2 $\quad b_k = 1$ and $p = 0$.

Here we allow patients to enter at a constant rate, and new waiting lists are created at a constant rate, but no patients are ever treated. In this situation the above equations can be solved exactly. Now Eq.(10) becomes

$$s_k = \frac{q}{r}\left[s_{k-1} - s_k\right] + r\delta_{k,0}. \tag{16}$$

which is easily solved to yield an exponential distribution of waiting lists

$$s_k = \frac{r^2}{q+r}\left[\frac{q}{q+r}\right]^k. \tag{17}$$

The number of patients waiting in the system diverges as $q \to \infty$. Whilst at first glance this model may seem unrealistic, it represents a situation that occurs in many healthcare systems when some sort of crisis occurs and in order to deal with this, no elective referrals take place. This can occur, for instance, when there is a local or national disaster or when there is a winter flu epidemic.

Case 3 $\quad a_k = k$ and $b_k = 1$ when $p = q = 1$ and $r = 0$.

In this situation the number of patients in the system is fixed on average, and does not grow as in the previous case. Thus we do not expect to recover a power-law. The rate equation is now given by

$$\frac{dn_k(t)}{dt} = \frac{1}{M(t)}\left[(k+1)n_{k+1} - kn_k\right] + \frac{1}{N(t)}\left[n_{k-1} - n_k\right] \tag{18}$$

where $N(t)$ and $M(t)$ are given in Eq.(7). In this case $n_k(t)$ and the moments $N(t)$ and $M(t)$ are independent of time. In the limit $t \to \infty$ it is straightforward to show that there is a stationary solution

$$n_k = N(0)\frac{\alpha^k}{k!}e^{-\alpha} \tag{19}$$

where $\alpha = M(0)/N(0)$. Thus in this case the length of the waiting lists obeys a Poisson distribution and we see that once the system stops growing continuously, and becomes effectively closed, the power-laws disappear.

Case 4 $\quad b_k = \frac{1}{k+1}$ and $r = 0$.

In this system patients join a waiting list with a rate inversely proportional to its length, which is the first case we have considered that directly models the preferential behaviour of patients. We consider two variants of this model. Firstly, a patient is chosen at random and removed from her list, so that $a_k = k$ and secondly a list is chosen at random and, if possible, a patient is removed from the list, so that $a_k = 1$ for $k > 0$ and $a_0 = 0$. We have performed a simulation of both these systems with 1000 waiting lists, $p = q = 1$ and initially 10 patients in each list. The results for the average value of n_k against k for 100 samples is shown in Fig. 1. We see that for both systems the distribution of the length of the waiting lists is very different from power-law, and looks more like a Gaussian. The mean value of k for $a_k = 1$ remains fixed at 10 patients per list whereas for $a_k = k$ the mean grows linearly with time.

Case 5 $\quad a_k = b_k = 1$ for $k > 0$ with $b_0 = 1$ and $r = 0$.

This merely represents a biased random walk along a semi-infinite chain in which $n_k(t)$ is proportional to the probability of finding the walker on site k at time t. For this reason we know that when $p > q$ the system evolves to a stationary distribution with

$$n_k(\infty) \sim (\frac{q}{p})^k \tag{20}$$

which is exponential. When $p < q$ the number of patients in the system grows with t and the mean length of the lists is $(q - p)t$.

III. DISCUSSION

In [2] the month to month variation in the length of the waiting lists of four dermatology specialists was measured over a six year period. The distribution of the fractional change λ in the length of the waiting lists, $b(\lambda)$, was found to be power-law. In the previous section we considered $n_k(t)$, the number of waiting lists of length k at time t. These quantities are related via the formula

$$b(\lambda) = \frac{1}{T}\sum_{t=1}^{T}\sum_{k=1}^{\infty}\sum_{k'=1}^{\infty} n_k(t) n_{k'}(t+1)\delta(\lambda - \frac{k}{k'}). \tag{21}$$

When the measurement period is very much longer than the time between successive measurements, $T >> 1$, and the distribution of waiting list lengths is only weakly dependent on time, then we can write

$$b(\lambda) = \sum_{k=1}^{\infty}\sum_{k'=1}^{\infty} n_k n_{k'}\delta(\lambda - \frac{k}{k'}). \tag{22}$$

Thus if we have a power-law distribution of waiting lists, $n_k \sim k^{-\beta}$, then it follows that the distribution of the fractional change is also power-law, $b(\lambda) \sim \lambda^{-\beta}$, with the same exponent.

If one assumes that these power-law distributions are generally valid for hospital waiting lists, then one can estimate the length of the longest waiting lists. In particular, if the number of lists of length k behaves as $n_k \sim k^{-\beta}$, then the number of lists m is related to the length of the maximum list, N_{max} by

$$N_{max} \sim m^{\frac{1}{\beta-1}}. \tag{23}$$

We can use this formula to estimate N_{max} and β from real data. In England and Wales there are 4349 surgical consultants, and adding the non-consultant surgeons, gives a total of, say, $m = 5000$ surgical waiting lists [10]. This suggests that the longest waiting list in England and Wales is 5000 patients, if the exponent $\beta = 2$ (Smethurst and Williams [2,9]), and 292 patients if $\beta = 2.5$ (Papadopoulos et al [3]). The longest reported waiting list we found was 1800 patients [11], which would imply that the exponent in the power-law is $\beta = 2.14$.

A number of variants to the above models are possible. For instance, we can introduce the possibility of patients hopping from list to list, or the possibility of the removal of waiting lists with a low average length. Similarly, we could give each list a different rate to allow patients to be treated at different rates in different lists. Whilst many of the these additions would make the models more realistic, they do not effect our overall conclusions. Namely, that power-laws can arise in some simple models of waiting lists, but when some type of preferential choice is introduced, there are no power-laws. This observation seems to be in accord with out basic understanding of when power-laws arise in complex systems. Normally, a power-law distribution indicates that extremal events are much more likely to occur than would be expected with an exponential or Gaussian distribution. Thus it seems unlikely that a power-law distribution of waiting lists will arise as a result of patients' reluctance to join long waiting lists. If power-laws do exist in hospital waiting lists, then the mechanisms creating them are probably indistinguishable from the random selection of lists by patients.

IV. ACKNOWLEDGEMENTS

GJR would like to thank the Leverhulme Trust for their financial support.

V. FIGURE CAPTION

1. A graph of n_k against k for Case 4 when $a_k = 1$ ($\diamond$) and $b_k = k$ ($+$).

VI. TABLE CAPTION

1. Summary of our models and the resulting distribution of waiting lists.

REFERENCES

[1] http://www.checklist.co.uk/.

[2] Smethurst, D. P. and Williams, H.C. Nature **410**, 652-653 (2001).

[3] Papadopolous, M. C. *et al*, J. Roy. Soc. Med. **94**, 613-616 (2001).

[4] Smethurst, D. P. and Williams, H.C. J. Roy. Soc. Med. **95** 287-289 (2002).

[5] Bak, P. *How Nature Works: The Science of Self-Organised Criticality* (Springer, New York 1996).

[6] Gross, G. and Harris, C. M. *Fundamentals of Queuing Theory* (Wiley, New York 1998).

[7] Kullmann, L. and Kertesz, J. Physica A **299** 121-126 (2001).

[8] Albert, R. and Barabasi, A.-L. Rev. Mod. Phys **74** 47-97 (2002).

[9] Freckleton, R. P. and Sutherland, W. J. Nature **413**, 382 (2001).

[10] Private Communication, R. Brackenbury, Royal College of Surgeons.

[11] http://society.guardian.co.uk/NHSstaff/story/0,7991,438272,00.html. Date accessed 18 March 2002.

Table 1

Case	a_k	b_k	p, q and r	List Distribution
1	k	1	$q > p$, $r = 0$	Power-law
2	-	1	$p = 0$	Exponential
3	k	1	$p = q$, $r = 0$	Poisson
4	1 and k	$\frac{1}{k+1}$	$r = 0$	Gaussian-like
5	1	1	$r = 0$	Exponential