Modelling Inflation as a Random Process

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Abstract

We introduce a model of inflation in which the prices of commodities are inflated by a random process. At each time step a price x is selected with a rate of x^{α} and is inflated by a factor of $1/\beta$ where $0 < \beta < 1$. For $\alpha = 0$, we obtain a general time-dependent solution, where the initial price distribution can be of any form. When $\alpha > 0$, in the long time limit, only the highest price inflates. For $\alpha < 0$, the model exhibits asymptotic scaling behaviour. We also consider the effects of a time dependent β, where $0 < \beta(t) < 1$, for the case $\alpha = 0$. We find that the price distribution approaches a steady state if $\beta(t) - 1 \sim 0$ faster than $1/t$.

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I. INTRODUCTION

Economic systems currently present some of the most fascinating challenges facing researchers of complex systems. In recent years physicists have made substantial progress in applying ideas from statistical physics to problems in economics, [1,2]. This is part of a tradition of applying ideas from physics to social problems, which stretches back to the nineteenth century [3].

For example, speculative markets, in which traders buy and sell for profit, rather than for consumption, represent an interesting sub-class of problems [4]. Examples of these markets include property, land, stock, commodity and book markets. The prices in these markets show *speculative episodes* or *peaks* [4] which can be caused by a number of factors including interest rates, supply and demand, inflation and even the outbreak of war. One characteristic common to many markets is the *price multiplier effect* which occurs during a speculative episode when the price of expensive items increases more than the price of less expensive items. One possible mechanism for this observation is that initially speculation is localised, either physically or by market sector, in a part of the market where prices are more expensive. This speculation drives up prices in the expensive part of the market, which then spreads to the less expensive part of the market.

This paper focuses on the effects of inflation on price distributions of commodities. Inflation can be described quantitatively in terms of an inflation rate which varies over time [5].

From the empirical study of price indices, it is well known that the exponential growth of price changes is a universal property of inflation. Recently, Mizuno and Takayasu [6] studied the related mechanism of double-exponential growth during hyper-inflation. Hyper-inflation can be thought of as very high inflation [7]. It was concluded that the double-exponential behaviour was caused by the effect of auto-catalytic shortening of the characteristic time.

The model introduced in this paper is one where prices are inflated by a simple random process. At each time step, a price x is chosen with a rate of x^{α} and is inflated by a factor of $1/\beta$ where $0 < \beta < 1$. The inflation rate will be determined by the value of β. Hence, a small β value will provide us with a high inflation rate. Through repeated inflation, a time-dependent price distribution is developed and we will study how this distribution depends on the parameters α and β. This approach is in contrast to that taken in say, [8], where a deterministic, non-random, incremental inflation mechanism is assumed.

Our model is a simplified one that focuses on just one mechanism that may influence the price of a commodity. However, inflation does play a significant role in the fluctuation of prices and the potential utility of our model is that it yields realistic price distributions. With a constant inflation rate β, this model has three distinct types of behaviour. When $\alpha = 0$, prices to be inflated will be chosen with an equal probability and the price distribution is time-dependent. When $\alpha > 0$, high priced commodities will have more chance of being picked for inflation and eventually the system evolves into a state in which only the single highest price is being inflated and all the other prices are stationary. When $\alpha < 0$, lower priced commodities will be more likely to have an increase in their price and the system has a scaling solution.

In the following section, we introduce our model and then go onto present the main theoretical results for each variant of the model. In the final section we consider the problem of a time-dependent β, and show how different price distributions are obtained depending on the speed with which $\beta(t)$ approaches 1 as $t \to \infty$.

II. THE MODEL

In this section, we consider prices inflated by a constant factor of $1/\beta$, at each time step, where $0 < \beta < 1$. A price x is chosen for inflation with a rate of x^α where α can take any value.

The rate equation for this model is

$$\frac{dc(z,t)}{dt} = \int_0^\infty c(x,t)f(z;x)dx - \int_0^\infty c(z,t)f(x;z)dx, \tag{1}$$

where $c(z,t)$ is the number of commodities with price z at time t and $f(z;x)$ is the rate of creating a price z from a price x via inflation. We choose

$$f(z;x) = x^{\alpha}\delta(z - x/\beta), \tag{2}$$

so that after substituting in $f(z;x)$ and integrating over the delta function we obtain

$$\frac{dc(z,t)}{dt} = \beta^{\alpha+1}z^{\alpha}c(\beta z,t) - z^{\alpha}c(z,t). \tag{3}$$

The first term on the right hand side of Eq.(3) accounts for the gain in $c(z)$ due to the inflation of a price $\beta z \rightarrow z$. The second term accounts for the loss in $c(z)$ due to the inflation of a price $z \rightarrow z/\beta$.

Case A $\alpha = 0$.

When $\alpha = 0$, Eq.(3) reduces to

$$\frac{dc(z,t)}{dt} = \beta c(\beta z,t) - c(z,t), \tag{4}$$

and it is evident that each price has an equal probability of being inflated by a factor of $1/\beta$.

To find the resulting price distribution, let us consider the moments $M(p,t)$, defined by

$$M(p,t) = \int_0^{\infty} z^p c(z,t)dz. \tag{5}$$

The moments $M(p,t)$ obey

$$\frac{dM(p,t)}{dt} = \left[\frac{1}{\beta^p} - 1\right] M(p,t), \tag{6}$$

with solution

$$M(p,t) = M(p,0)exp\left[(\frac{1}{\beta^p} - 1)t\right]. \tag{7}$$

If the initial price distribution is $c(z) = e^{-z}$ at $t = 0$, then

$$M(p,t) = \Gamma(p+1)exp\left[(\frac{1}{\beta^p} - 1)t\right] = \Gamma(p+1)e^{-t}\sum_{r=0}^{\infty}\frac{t^r}{r!}\frac{1}{\beta^{pr}}. \tag{8}$$

By using the fact that if $c(z,t) = \gamma e^{-\gamma z}$ then $M(p,t) = \frac{1}{\gamma^p}\Gamma(p+1)$, and applying this term by term to the right hand side of Eq.(8) with $\gamma = \beta^r$, it is simple to show that

$$c(z,t) = e^{-t} \sum_{r=0}^{\infty} \frac{(\beta t)^r}{r!} e^{-z\beta^r}. \tag{9}$$

It is obvious from Eq.(9) that the price distribution does not reach a steady state.

One can easily verify that the first moment, the total commodity density, is conserved, $M(0,t) = 1$.

A more general solution can also be determined for arbitrary $c(z,0)$. In this case, the resulting price distribution is

$$c(z,t) = e^{-t} \sum_{r=0}^{\infty} \frac{(\beta t)^r}{r!} c(z\beta^r, 0). \tag{10}$$

Thus whatever the form of the initial price distribution, the price distribution never approaches a steady state.

Case B $\alpha > 0$.

In this case, the highest priced commodities will be chosen for inflation with a greater rate. Initially, a limited number of low priced commodities and a greater number of high priced commodities will inflate by $1/\beta$. After inflation, the prices, and their probability of being picked for inflation will increase simultaneously.

In the large-time limit, this will lead to only one of the initial highest prices inflating and dominating the overall price distribution. As this one high price progresses with time, it will be inflating at a rate of β^{-t}. At long times, none of the other prices will change, and the rest of the distribution will become stationary.

Consequently the resulting overall price distribution will no longer be continuous but instead made up of a single continuous stationary piece and a delta function spike at a position moving like β^{-t}.

To further illustrate this point, consider a model in which there are only two prices, x and y. In this case $x \to x/\beta$ and $y \to y$ with probability $x^\alpha/(x^\alpha + y^\alpha)$ and vice versa. If we take z to be the ratio of the two prices, $z = x/y$, then $z \to z/\beta$ with probability $z^\alpha/(z^\alpha + 1)$

and $z \to \beta z$ with probability $1/(z^\alpha + 1)$. It is evident that through an iterative process, the two prices become separated exponentially. The average value of z obeys

$$z' = \frac{z^{\alpha+1}}{\beta(z^\alpha + 1)} + \frac{\beta z}{z^\alpha + 1}. \tag{11}$$

Initially, it is likely that both prices will inflate, hence z will flow between 0 and ∞ due the increase in one of the two prices. Eventually, either $z \to 0$ or $z \to \infty$ as $t \to \infty$ and one of the terms on the right hand side of Eq.(11) will go to 0. Thus one of the two prices will always be chosen for inflation, and will grow relative to the other price.

Case C $\alpha < 0$.

When $\alpha < 0$, Eq.(3), yields the equation for moments

$$\frac{dM(p,t)}{dt} = \left[\frac{1}{\beta^p} - 1\right] M(p+\alpha, t). \tag{12}$$

Asymptotically, the moments exhibit a power-law behaviour

$$M(p,t) \sim a_p t^{b_p} \tag{13}$$

as $t \to \infty$. Inserting this form into Eq.(12) gives

$$b_p = -\frac{p}{\alpha} \tag{14}$$

and hence

$$M(p,t) \sim t^{-\frac{p}{\alpha}}. \tag{15}$$

Eq.(13) implies that in the long-time limit, $c(z,t)$ approaches the scaling form,

$$c(z,t) = \frac{1}{z} f(zt^{1/\alpha}), \tag{16}$$

where the scaling function $f(u)$ obeys

$$\frac{1}{\alpha} f'(u) = u^{\alpha-1}[\beta^\alpha f(\beta u) - f(u)]. \tag{17}$$

It is simple to show that as $u \to 0$ then $f(u) \sim e^{-u^\alpha}$ while as $u \to \infty$, $f(u)$ approaches a constant. Furthermore, the total number of commodities $N(t)$, $N(t) \equiv M(0,t)$, is conserved, while the typical price size $\langle z \rangle$, $\langle z \rangle = M(1,t)/N(t)$, grows as $t^{-1/\alpha}$.

The mechanism for this scaling form is simply that the rate of lower prices being selected for inflation is extremely high in comparison to the larger prices. Hence the distribution of prices evolves as a continuous distribution.

III. A TIME DEPENDENT β

The above work concentrates on a β-value which remains constant over time. However a number of studies have concentrated on identifying the time dependent nature of inflation [5]. Consequently we now consider a time dependent β where $0 < \beta(t) < 1$ and as $t \to \infty$, $\beta(t) \to \beta$. If $\beta \neq 1$ then the results of the previous section hold, and the system behaves as if $\beta(t)$ was independent of time, and had a fixed value β. However if $\beta(t) \to 1$ as $t \to \infty$, then new behaviour is possible, and we consider this case in the rest of this section.

For $\alpha = 0$, we have already established that the moments $M(p,t)$ obey Eq.(6), and when the initial distribution is given as the function $c(z) = e^{-z}$ at $t = 0$, a general solution is

$$M(p,t) = \Gamma(p+1) exp\left[\int_0^t (1/\beta(t)^p - 1)dt\right]. \tag{18}$$

We consider $\beta(t)$ such that

$$\beta(t) = 1 - \frac{\lambda}{(t+1)^\nu}, \tag{19}$$

where $0 < \lambda < 1$ and $\nu > 0$. In this case it is possible to have high inflation initially, depending on the parameters λ and ν, and in the long-time limit, the inflation rate will reach 1. In other words there is no inflation as $t \to \infty$.

To determine the moments for this $\beta(t)$ form, we need to consider the function

$$I(t) = \int_0^t \left[\frac{1}{(\beta(\tau))^p} - 1\right] d\tau. \tag{20}$$

Substituting Eq.(19) into Eq.(20) and taking the limit $t \to \infty$ gives

$$I(t) = \int_0^t \left[\left(1 - \frac{\lambda}{(\tau+1)^\nu}\right)^{-p} - 1\right] d\tau \to p\lambda t \int_0^1 (tu+1)^{-\nu} du \tag{21}$$

where the second integral is obtained by changing variables, $u = \tau/t$ before taking the large t limit. It is evident that there are three distinct types of behaviour for different ν values. When $0 < \nu < 1$, we find

$$I(t) = \frac{p\lambda}{1-\nu} t^{1-\nu}, \tag{22}$$

hence by entering this $I(t)$ into Eq.(18), we find that

$$M(p,t) = \Gamma(p+1) exp\left[\frac{p\lambda}{1-\nu} t^{1-\nu}\right] \tag{23}$$

and the solution for the price distribution is

$$c(z,t) = exp\left[\frac{\lambda}{\nu-1} t^{1-\nu} - zexp(\frac{\lambda}{\nu-1} t^{1-\nu})\right]. \tag{24}$$

Using the same approach, we find that when $\nu = 1$, the moments and resulting price distribution are given by

$$M(p,t) = \Gamma(p+1)(t+1)^{p\lambda} \qquad \text{and} \qquad c(z,t) = (t+1)^{-\lambda} exp\left[-z(t+1)^{-\lambda}\right]. \tag{25}$$

For $\nu > 1$, we find that the eventual price distribution is time independent. The moments are given by

$$M(p,\infty) = \Gamma(p+1) exp\left[\frac{p\lambda}{\nu-1}\right], \tag{26}$$

hence the steady state price distribution is

$$c(z,\infty) = exp\left[\frac{\lambda}{1-\nu} - zexp(\frac{\lambda}{1-\nu})\right]. \tag{27}$$

Thus, as $t \to \infty$, if $\beta(t) - 1 \sim 0$ faster than $1/t$ then the resulting price distribution is of a time independent exponential form. On the other hand, if $\beta(t) - 1 \sim 0$ slower than $1/t$ then the eventual price distribution would exhibit time dependence.

IV. DISCUSSION

We have presented a model of price inflation in which prices of commodities are chosen for inflation with a specific rate. We have also investigated the behaviour of price distributions for these distinct rates. It has been established that the rate of choosing a price for inflation has influenced the price distribution of a population of commodities.

The model has three variants. One case is where each price has an equal chance of being picked for inflation. In this case, for arbitrary initial conditions, the price distribution remains time-dependent and never reaches a steady state.

We next considered the variant in which larger prices had a higher rate of being chosen for inflation. It was evident that in the large-time limit, only one of the initial high prices would increase. This is explained by the fact that both the high price and its rate for being picked for inflation would increase simultaneously, hence the remaining prices wouldn't have the opportunity of being selected for inflation.

For the verion of the model in which smaller prices were more likely to be picked for inflation, an exact solution was not obtained. However, it was found that the price distribution approached a scaling form. It would be intersting to see what, if any, scaling was observed in distribution of prices taken from a real economic system.

In addition, we inserted a time dependent β which approached 1 as $t \to \infty$ for the case $\alpha = 0$. We found that if $\beta(t) - 1 \sim 0$ faster than $1/t$ then it was possible for the price distribution to reach a steady state. If β approached $\beta \neq 1$ as $t \to \infty$ then in the long time limit the system behaves as if $\beta(t)$ were time independent.

In conclusion, we have found that the rate of choosing prices for inflation in this random process can have a great effect on the distribution of prices. It would be interesting to try and quantify the extent to which inflation in a real economic system depended on random processes of this type rather than the more orthodox deterministic incremental price rises.

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