Growth and Coagulation in a Herding Model

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Abstract

We discuss various existing models which mimic the herding effect in financial markets and introduce a new model of herding which incorporates both growth and coagulation. In this model, at each time step either (i) with probability p the system grows through the introduction of a new agent or (ii) with probability $q = 1 - p$ two groups are selected at random and coagulate. We show that the size distribution of these groups has a power law tail with an exponential cut-off. A variant of our basic model is also discussed where rates are proportional to the size of a group.

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I. INTRODUCTION

From the empirical analysis of financial price-data on short time scales, it appears that the price variations of many assets, indices and currencies deviate from the Gaussian distribution and have fat-tails (see Mandelbrot, 1963; Mantegna and Stanley, 2000; Amaral et al., 2000). According to Pictet et al. (1997), exceptionally large fluctuations are distributed according to an exponentially truncated power law, where

$$P(x) \sim \frac{A}{x^{\mu+1}} exp\left[-\frac{x}{x_0}\right] \tag{1}$$

in which μ takes a value between 1.4 and 1.6 for a number of market indices and stocks. It is believed that this behaviour is brought about by a herding effect.

Bikhchandani and Sharma (2001) and Kim et al. (2003) describe the term herding by a group of agents sharing the same information or the same rumour and imitating each other when making decisions. An agent may imitate others if he believes that they have reliable information or knowledge about the market. Devenow and Welch (1996), stated that there are two polar views of herding; these being the non-rational and the rational views. The non-rational view is considered as agents following each other blindly or behaving like lemmings and the rational view focuses on there being imperfect information (information-based herding), concern for reputation (reputation-based herding) and compensation structures (compensation-based herding) (see Bikhchandani and Sharma).

Information-based herding can be described by an investor having private (but imperfect) information about what trading decision to make. This private information may be obtained from research carried out by the investor. However, instead of making their own choice, they ignore their private information and follow the actions of prior investors (see Bikhchandani and Sharma; Devenow and Welch; Wermers, 1999). For example, an investor may have private information which leads him to believe that it is not wise to purchase, though he may be tempted if he sees that four investors have previously purchased. In this case he may disregard his private information and imitate the others.

Scharfstein and Stein (1990) and Wermers believe that reputation-based herding occurs when an agent follows the actions of others to gain or maintain reputation. Therefore managers, for example, who invest on behalf of others, may be reluctant to act according to their own beliefs and private information as they may be concerened about how others will assess their ability to make sound judgements, which may in turn damage their reputation. For example, in the stock market, if prices are low then investors' may be hesitant to buy in case prices were to decrease further. However, in the more likely event of a price increase, they wouldn't look so bad as they all would have suffered the same fate. Therefore in order to preserve reputation, professional managers may *follow the herd*.

Compensation-based herding occurs when an investors' compensation depends on how their performance compares to other investors' performance (see Bikhchandani and Sharma). Hence similar actions between agents may give a higher compensation, which leads to this herding behaviour.

These groups or herds of agents are thought to be responsible for the large fluctuations in prices. Herding coupled with supply and demand can cause sudden rises or falls when the groups all decide to buy/sell. The large groups have a strong impact on the supply and demand equilibrium, hence the larger the group, the bigger the impact they have on the market.

Several models have been proposed recently to mimic and investigate the herding behaviour in financial markets (see, e.g. Cont and Bouchaud, 2000; Eguíluz and Zimmermann, 2000; D'Hulst and Rodgers, 2000a, 2000b, 2001; Rodgers and Zheng, 2002; Rodgers and Yap, 2002). In order to describe these models, it is important to understand the meaning of a cluster and how this relates to the distribution of returns. If we consider a set of M agents, then an agent can be connected to any of the $M-1$ other agents. Hence a cluster is a connected group of agents who can exchange information and make the same decisions from a choice of buying, selling or doing nothing. If we assume that the traded amount is proportional to the number of agents in a cluster (see Eguíluz and Zimmermann), then the cluster size distribution can correspond to the distribution of returns *i.e.* the relative price variation on a given time interval. The return $R(t)$, at time t, in this case, can be obtained by generating the difference between the number of buyers and the number of sellers, or alternatively, the supply and demand balance (see Rosenberg and Ohlson, 1976), at a particular time. If agents are buying, the return is positive and it is negative if agents are selling. Cont and Bouchaud assume that $R(t)$ is proportional to the change in the logarithmic of the price, that is

$$\frac{R(t)}{\lambda} = \ln P(t) - \ln P(t-1) \tag{2}$$

where $P(t)$ is the price at time t and λ is a factor measuring the market liquidity or the sensitivity of price to fluctuations in demand. If n_s is the number of clusters of size s, for a system with M agents, then the probability to have a return of size s/M is given by sn_s/M.

Cont and Bouchaud introduced a model in which the network of agents was considered to be a diluted regular lattice where the vertices were agents and edges were channels of communication. In this model, any agents were connected with probability p and agents in a connected cluster formed a group which shared information and made the same decisions. A power law distribution with an exponential cut-off of group sizes, which corresponds to a distribution of returns, was obtained when the parameter p was close to the percolation threshold.

To further investigate the herding effect, Eguíluz and Zimmermann put forward a kinetic version of the above model in which, instead of being static, the network of agents evolved dynamically as decisions were being made or as agents exchanged information. They obtained numerically the same power law distribution for returns as Cont and Bouchaud. An exact analytical expression was found for this model by D'Hulst and Rodgers (2000b) and since then alternative kinetic models have been introduced by Rodgers and Zheng and Rodgers and Yap. These three models (see D'Hulst and Rodgers, 2000b; Rodgers and Zheng; Rodgers and Yap) are very closely related as they incorporate only two phenomena from either growth, fragmentation, coagulation, addition, or attachment; however it is the combination of these ingredients which cause the models to differ. In terms of clusters, *growth* is the process in which a new cluster of size 1, or a free agent, is introduced to the system which allows it to expand.

Fragmentation can be defined as the process in which a cluster breaks up into isolated agents. To further illustrate this, if an agent is selected at random, they may decide to trade (buy or sell) and all agents in the same cluster would share their information and take the same trading decision. The cluster would then fragment into individual agents.

Coagulation is the process in which two clusters join. So if an agent decides to do nothing, apart from taking the opportunity to gossip (network), then another agent is selected at random and these two agents are connected thereby making one cluster from two. *Addition* can be described as a special type of coagulation. It is the process in which a free agent already in the system is added at random to another cluster. Hence it can only occur when the system consists of free agents.

Attachment is the process in which a new incoming agent joins an existing group, or, attaches themselves to an existing group. This in turn allows the system and the clusters to grow in size.

The model proposed by D'Hulst and Rodgers (2000b) is one where groups of agents can either coagulate or fragment at each time step. The model introduced by Rodgers and Zheng is a combination of attachment and fragmentation and the third model (see Rodgers and Yap) incorporates both growth and addition.

In this paper, we introduce an alternative kinetic model in which either the system grows through the introduction of a new agent or two groups of agents coagulate. In Section 2 we review the three existing kinetic models (see D'Hulst and Rodgers, 2000b; Rodgers and Zheng; Rodgers and Yap) and the main results obtained. In Section 3, we present our main model and show that the size distribution of groups of agents has a power law tail with an exponential cut-off. We assume that rates are independent of group sizes. In Section 4, we study a variant of the model in which groups are selected for coagulation with a rate proportional to group sizes. In this case large groups grow more quickly than small ones. This seems reasonable as it is more likely that two agents from larger groups would come into contact with each other. In the final section we discuss our work and draw some conclusions.

II. EXISTING MODELS

Model A *Fragmentation and Coagulation*

In the model proposed by D'Hulst and Rodgers (2000b), with probability p a cluster is selected at random for fragmentation and with a probability $1-p$ two clusters are selected at random and coagulated. As a chemical process this can be written as $B_r \rightarrow rB_1$ with probability p and $B_r + B_s \rightarrow B_{r+s}$ with probability $1-p$. All rates are proportional to group sizes.

This model was solved exactly and it was found that n_s which is the number of clusters of size s, obeys

$$n_s = \frac{(1-p)^{s-1}(2s-2)!}{(2-p)^{2s-1}s!^2} N_0 \tag{3}$$

where N_0 is the total number of agents. For large s, it was found that

$$n_s \sim N_0 \left(\frac{4(1-p)}{(2-p)^2} \right)^s s^{-5/2} \tag{4}$$

which is power law with an exponent 5/2 with an exponential cut-off that vanishes when $p = 0$. This shows that the model is not self-organised as a power-law distribution for large transactions only appears when the parameter p is tuned to a low value.

The variance of the distribution of returns, σ^2, which is given by

$$\sigma^2 = M_3 - M_2^2 = \frac{(1-p)(2-p)}{p^3} \tag{5}$$

where

$$M_i \equiv \frac{1}{N_0} \sum_{s=1}^{\infty} s^i n_s \tag{6}$$

is the ith moment of n_s, was obtained. The kurtosis was also computed, which indicates that there are heavy tails for every value of p. This shows that the model deviates from Gaussian behaviour.

Model B *Attachment and Fragmentation*

In the model introduced by Rodgers and Zheng, at each time step, with probability p an incoming agent joins an existing cluster of size k with a rate proportional to k and with probability $1-p$ a cluster is selected at random and fragmented. As a chemical process this can be written as $B_1 + B_r \to B_{r+1}$ with probability p and $B_r \to rB_1$ with probability $1-p$.

In this model, for large t, $n_k(t)$ which is the number of clusters with size k, is

$$n_k(t) = \frac{p(1-p)}{\alpha}\Gamma(\beta)\frac{\Gamma(k)}{\Gamma(k+\beta)}t. \tag{7}$$

As $k \to \infty$,

$$n_k(t) \sim tk^{-\beta} \tag{8}$$

where

$$\beta = 2 + \frac{1-p}{\alpha} = 2\left[\frac{\sqrt{4p/(1-p)+1}}{\sqrt{4p/(1-p)+1}-1}\right]. \tag{9}$$

Hence this model gives a power law distribution of groups of agents for all parameter values, with an exponent which can take any value greater than 2.

Model C *Growth and Addition*

In the model presented by Rodgers and Yap, at each time step, with probability p the system grows through the introduction of a new agent and with probability $1-p$ a free agent already in the system is added at random to a group of size k with rate k. As a chemical process this can be written as $0 \to B_1$ with probability p and $B_r + B_1 \to B_{r+1}$ with probability $1-p$.

By using a similar method to that of the above model, it was found that when $p > 1/2$

$$n_k(t) = p(2p-1)\Gamma\left(1+\frac{1}{1-p}\right)\frac{\Gamma(k)}{\Gamma(k+\frac{1}{1-p})}t \tag{10}$$

where as $k \to \infty$,

$$n_k(t) \sim tk^{-\frac{1}{1-p}}. \tag{11}$$

In summary, exact analytical expressions were obtained for the above models. It was found that whilst one has a power law distribution for group sizes with a tuneable exponent, others have exponents that are determined by the model as universal exponents.

III. A MODEL WITH GROWTH AND COAGULATION

We introduce a model in which at each time step one of the two events, chosen at random, can occur. With probability p an agent joins the group but remains free. Alternatively, with probability $1-p$, two groups are selected for coagulation with a rate independent to group sizes. As a chemical process this can be written as $0 \to B_1$ with probability p and $B_r + B_s \to B_{r+s}$ with probability $1-p$. Consequently the number $n_s(t)$ of clusters of size s at time t evolves like

$$\frac{dn_s(t)}{dt} = \frac{q}{N(t)^2}\sum_{r=1}^{s-1} n_r n_{s-r} - \frac{2qn_s}{N(t)} + p\delta_{s,1}. \tag{12}$$

In this equation

$$N(t) = \sum_{s=1}^{\infty} n_s(t) \tag{13}$$

represents the number of groups and

$$M(t) = \sum_{s=1}^{\infty} sn_s(t) \tag{14}$$

is the number of agents in the system. The first term on the right-hand side of Eq. (12) describes the creation of a new cluster of size s by coagulation of two clusters of size r and $s-r$, with $r < s$. The second term describes the disappearance of a cluster of size s by coagulation with another cluster and the final term represents the arrival of new agents in the system. Using rate equation (12) it is a simple matter to show that

$$\frac{dN(t)}{dt} = 2p - 1 \tag{15}$$

and

$$\frac{dM(t)}{dt} = p. \tag{16}$$

Eq. (15) represents the fact that with probability $2p-1$, *on average*, the number of groups increases by 1. Similarly, Eq. (16) indicates that with probability p the number of agents increases by 1. The form of Eqs. (12), (15) and (16) suggests that the solution for $n_s(t)$, for $s = 1, 2, ...$, is linear in time for large t when $p > 1/2$. In this limit we can solve Eqs. (15) and (16) to yield

$$N(t) = (2p-1)t \qquad \text{and} \qquad M(t) = pt. \tag{17}$$

Writing

$$n_s(t) = tc_s \tag{18}$$

we find that

$$c_s = \frac{1-p}{(2p-1)^2}\sum_{r=1}^{s-1} c_r c_{s-r} - \frac{2(1-p)c_s}{2p-1} + p\delta_{s,1}. \tag{19}$$

Introducing the generating function

$$g(\omega) = \sum_{s=1}^{\infty} c_s \omega^s \tag{20}$$

we find

$$g(\omega) = \frac{2p-1}{2(1-p)}\left[1 - \sqrt{1-4(1-p)p\omega}\right], \tag{21}$$

giving a size distribution for clusters of

$$c_s = \frac{(2p-1)p^s(1-p)^{s-1}(2s-2)!}{s!(s-1)!}. \tag{22}$$

Using Stirling's formula to expand this for large s, we have

$$c_s \sim (4p(1-p))^s s^{-3/2}. \tag{23}$$

Hence this model presents a power law distribution with an exponent 3/2 for the size distribution of clusters, with an exponential cut-off. When $p \to 1/2$, the exponential cut-off vanishes.

IV. SIZE DEPENDENT RATES

We can introduce a version of the above model in which the rates are proportional to the size of a group. In this case the rate equation for the system is

$$\frac{dn_s(t)}{dt} = \frac{q}{M(t)^2}\sum_{r=1}^{s-1} rn_r(s-r)n_{s-r} - \frac{2qsn_s}{M(t)} + p\delta_{s,1}. \tag{24}$$

Using a similar method to that used in the previous section, but with an alternative generating function, it is simple to show that for large s

$$c_s \sim \left(\frac{p}{1-p}\right)^s s^{-5/2}. \tag{25}$$

Thus, although we do not find an exact analytical solution for this variant of the model, we find that large groups are approximately power law distributed with an exponential cut-off with an exponent 5/2. As $p \to 1/2$, the exponential cut-off vanishes.

V. DISCUSSION

We have discussed various different kinetic models of herding (see D'Hulst and Rodgers, 2000b; Rodgers and Zheng; Rodgers and Yap) and have introduced an alternative kinetic model which incorporates both growth and coagulation. The system grows as new agents are introduced and coagulation allows groups of agents to be formed. This is in contrast to the models proposed by Eguíluz and Zimmermann and D'Hulst and Rodgers (2000b), where there are a fixed number of agents.

We have presented an exact solution to this model and found that it exhibits a power law distribution with an exponential cut-off for the size distribution of clusters and hence for the distribution of returns. The model is not self-organised as the parameter p has to be close to 1/2 to ensure the appearance of groups of all sizes and only when p is close to this value, the system displays a power law distribution for large groups where the exponent is 3/2.

We have also considered a variant of the model where rates are proportional to the size of a group. Although we do not obtain an exact solution, we find that for large s only, the system displays a power law distribution with an exponential cut-off. In this case, when $p \rightarrow 1/2$, the system gives a power law distribution for large groups but with an exponent $5/2$.

These results agree to those found by Cont and Bouchaud, Eguíluz and Zimmermann, and D'Hulst and Rodgers (2000b) where the models exhibit a power law distribution of returns with an exponential cut-off and in order to obtain a power law, the parameters require tuning. However, other models (see Rodgers and Zheng; Rodgers and Yap) exhibit a power law distribution of returns for all its parameter values and the exponent of the power law distribution, in each model, can be varied continuously as it is dependent on the parameter p. Hence, whilst some models have a power law distribution of group sizes with a tuneable exponent, others have exponents that are determined by the model as universal exponents.

In conclusion, there seems to be a number of different processes that can give a power law distribution for group sizes. It seems as though the precise details of the processes are unimportant and the essential, universal, important ingredient for these models is the mean-field, long ranged interaction nature of all these systems. This can be related to the modern, highly connected world we live in, where social interactions are also considered to be mean-field and non-local in character. Models of this type can be expected to mimic simple processes in social networks, financial and economic phenomena where individuals share information by forming groups and break up when the information has been used. Processes for which this approach can be used range from information or rumour spreading, particularly in financial markets, through the take up of the latest craze, such as video phones or The Lord of the Rings, to epidemiological studies of disease and epidemic spread.

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