Applications of Random Matrix Theory in Lattice QCD

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Our group

our PhD student
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& myself
Outline

- Introduction to Lattice QCD
- Two-dimensional naive Discretization
- Wilson RMT
- What has the future in store for us?
Introduction to Lattice QCD

A high-energy electron on collision course with ...

... a quark, confined in the proton.

Action of continuum QCD

The partition function of $N_f$ fermionic flavors

$$Z = \int \exp \left[ -S_{\text{YM}}(A) - \sum_{j=1}^{N_f} \int \bar{\psi}_j (iD(A) - m_j) \psi_j \, d^4x \right] D[A, \psi]$$

The Yang-Mills action of SU(3)

$$S_{\text{YM}}(A) = \frac{1}{4g^2} \int \text{tr} \, F_{\mu\nu} F^{\mu\nu} \, d^4x$$

with the field strength tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + g [A_\mu, A_\nu]$$

The four components of the vector potential $A_\mu \in \text{su}(3)$ are $3 \times 3$ matrix valued functions.
The continuum Dirac-operator

Fermionic fields $\psi_j$ are Grassmann variables

$$ \Rightarrow Z = \int \prod_{j=1}^{N_f} \det (\gamma D(A) - m_j) \exp \left[ - S_{YM}(A) \right] D[A] $$

The Dirac operator

$$ D(A) = \gamma^\mu \left( \frac{1}{i} \partial_\mu + gA_\mu \right) $$

Index-theorem:
number of zero modes (index)=topological charge

$$ \nu = \frac{1}{32\pi^2} \int \varepsilon^{\mu\nu\lambda\kappa} \text{tr} F_{\mu\nu} F_{\lambda\kappa} d^4 x $$
QCD in Continuum

image by Derek Leinweber (CSSM)

image from www.llnl.gov

Lattice QCD
Lattice QCD

- space-time $V$ becomes discrete with lattice spacing $a$
- vector field $A_\mu \in su(3)$ replaced by $U_\mu \in SU(3)$

Big question:
How do we perform the limits $a \to 0$ and $V \to \infty$ such that we obtain continuum QCD?

volume of space-time: $V=L^4$
Fundamental problem on the lattice

**Energy in continuum:**

\[ E^2 = k_\mu k^\mu + M_0^2 \]

**Energy on lattice:**

\[ E^2 = \frac{\sin^2(k_\mu a)}{a^2} + M_0^2 \]

**Doubler Problem:**

\[ k_\mu \rightarrow \left\{ \begin{array}{l} \pi \frac{k_\mu}{a} - k_\mu \\ \end{array} \right. \]

one momentum\( = (2^4 = 16) \) particles
Many ways and no guide

Wilson Fermions
Staggered Fermions
Overlap Fermions
Domain Wall Fermions
The $\epsilon$-regime of QCD

- infrared limit of QCD
- large Compton wavelength of Goldstone bosons $\gg$ box size $V^{1/4} = L$
- lattice volume (space-time volume) $V \to \infty$

Saddlepoint approximation:
- spontaneous breaking of chiral symmetry
- global Goldstone bosons = Mesons

\( N_f = 2, SU(2) \text{-integral} = \text{zero momentum modes of the three pions} \)
Partition function in the $\epsilon$-regime for $N_f$ flavors

\[ Z \propto \int_{SU(N_f)} \exp[\mathcal{L}(U)] d\mu(U) \]
\[ \propto \sum_{\nu \in \mathbb{Z}} \int_{U(N_f)} \exp[\mathcal{L}(U)] \det^\nu U d\mu(U) \]

Lagrangian of the Goldstone bosons:

\[ \mathcal{L}(U) = \frac{\Sigma V}{2} \text{tr} M(U + U^\dagger) + \mathcal{L}_{\text{correction}}(V, a, U) \]

- index of the Dirac operator: $\nu$
- masses of the quarks: $M$
- low energy constants: $\Sigma$, …
... and the same for RMT

Model: \( D_{\text{QCD}} \rightarrow D = \begin{bmatrix} 0 & W \\ -W^\dagger & 0 \end{bmatrix} + D_{\text{correction}}(a) \)

\[
Z \propto \sum_{\nu \in \mathbb{Z}} \int_U \exp[\mathcal{L}(U)] \det^\nu U d\mu(U)
\]

Lagrangian of the Goldstone bosons:

\[
\mathcal{L}(U) = \frac{\Sigma V}{2} \text{tr} M(U + U^\dagger) + \mathcal{L}_{\text{correction}}(V, a, U)
\]

Derived by Shuryak and Verbaarschot (90's)!
Two-dimensional
naive Discretization

THE FAR SIDE

By GARY LARSON

"Ohhhhhhh... Look at that, Schuster...
Dogs are so cute when they try to comprehend
quantum mechanics."

3-23 Lucent *Carnegie Features, 1986
The naive Dirac Operator

Naive Dirac operator:

\[ D_{\text{naive}} = \frac{1}{2a} \gamma^\mu (T_\mu - T_\mu^{-1}) \]

Translation operator:

\[ T_\mu = T_\mu(U_{ij}^{\mu}) \]

- has the doubler problem
- **but** is the starting point for constructing staggered fermions
Why naive fermions?

- starting point for deriving staggered fermions
- same universality class as staggered fermions
- RMT model for staggered fermions by Osborn (2004), immensely complicated
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- starting point for deriving staggered fermions
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Why 2-D?

- simpler to understand
- our group has no supercomputer → numerically cheaper
Artificial chiral structure

General RMT model: \( D = \begin{bmatrix} 0 & W \\ -W^\dagger & 0 \end{bmatrix} \)

Original Classification (Verbaarschot, 90’s):

\( W \) is \( \begin{cases} 
\text{real,} & \beta = 1 \\
\text{complex,} & \beta = 2 \\
\text{quaternion,} & \beta = 4
\end{cases} \)
Artificial chiral structure

General RMT model: \( D = \begin{bmatrix} 0 & W \\ -W^\dagger & 0 \end{bmatrix} \)

Original Classification (Verbaarschot, 90’s):
\[
W \begin{cases} 
\text{real, } \beta = 1 \\
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\text{quaternion, } \beta = 4 
\end{cases}
\]

Reasons:
other dimensions = other universality classes

Artificial symmetry:
\[
\Gamma_x^5 T_x \Gamma_x^5 = -T_x, \quad \Gamma_y^5 T_y \Gamma_y^5 = T_y
\]
⇒ change of the universality class

Extensions into Altland-Zirnbauer classification!

Similar to the classification of topological insulators
(Schnyder, Ryu, Furusaki, Ludwig (2008))
Comparison: Lattice Data ↔ RMT
Three colors ($SU(3)$) & adjoint representation

odd-odd = 2-d cont. QCD

odd-even = level density of anti-symm. matrices

even-even = staggered fermions in 2-d
\[ \beta = 1 \] in 4-d cont. QCD

Drastic change of the universality class!
Wilson RMT

Kenneth G. Wilson
Main idea to solve the doubler problem:

- Make 15 particles infinitely heavy in the continuum limit \((a \to 0)\).
- too inertial, decouple from the system
- Wilson-Dirac operator

\[
D_W = D_{\text{naive}} + a\Delta
\]

\[
\propto \gamma^\mu \sin(k_\mu a) + \frac{\sin^2(k_\mu a/2)}{a}
\]

Laplace operator \(\Delta\)

- additional effective mass
- explicitly breaks chiral symmetry
- Dirac operator \(\gamma_5\)-Hermitian: \(D_5 = \gamma_5 D_W\) is Hermitian
Wilson RMT

Dirac operator:

\[ D_{\text{QCD}} \rightarrow D_W = \left( \begin{array}{cc} aA & W \\ -W^\dagger & aB \end{array} \right) + am_61 + a\lambda_7\gamma_5 \]

Weight:

\[ \exp[-S_{\text{YM}}] \rightarrow P(D_W) : \text{Gaussian} \]

- Hermitian random matrices \( A (n \times n), B ((n + \nu) \times (n + \nu)) \)
- and scalar random variables \( m_6, \lambda_7 \) are the Wilson-terms
  \( \Rightarrow \) breaking of chiral symmetry
- complex \( W (n \times (n + \nu)) \) matrix

Damgaard, Splittorff, Verbaarschot (2010)
Partition Function of $N_f$ flavors

$$Z \propto \int_{U (N_f)} \exp[\mathcal{L}(U)] \det^{\nu} U d\mu(U)$$

Lagrangian of the Goldstone bosons:

$$\mathcal{L}(U) = \frac{V\Sigma}{2} \text{tr} M(U + U^\dagger)$$

$$+ a^2 VW_6 \text{tr}^2 (U + U^\dagger) + a^2 VW_7 \text{tr}^2 (U - U^\dagger) + a^2 VW_8 \text{tr} (U^2 + U^{\dagger 2})$$

Damgaard, Splittorff, Verbaarschot (2010)

What are the low energy constants $\Sigma$ and $W_6/7/8$?
Effect of the low energy constants

Effect of $W_6$

$4 \hat{a}_6$

Effect of $W_7$

$4 \hat{a}_7$
Please do not read this now!

Will published soon!
We are comparing this to lattice data right now!
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Comparison with lattice data

- integrated distributions of individual eigenvalues of $D_5 = \gamma_5 D_W$
- Deuzman, Wenger, Wuilloid (2011)

- level density of real eigenvalues of $D_W$
- Damgaard, Heller, Splittorff (2012)
What has the future in store for us?

image from libertyscientist.com
In the 90’s:

- chiral RMT in QCD
- Shuryak, Verbaarschot

In the 00’s:

- chiral Ginibre RMT in QCD at finite chemical potential and temperature
- Akemann, Damgaard, Osborn, Splittorff, Verbaarschot, Wettig et al

In the 10’s:

- broken chiral RMT in lattice QCD
- Akemann, Damgaard, Kieburg, Osborn, Splittorff, Verbaarschot, Zafeiropoulos

Future:

- broken chiral RMT in lattice QCD at finite chemical potential and temperature
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Future:
▶ broken chiral RMT in lattice QCD at finite chemical potential and temperature
Be ready for the next round in RMT for QCD!

image from chessbase.de
Thank you for your attention!

Some papers:

- Kieburg, Splittorff, Verbaarschot: arXiv:1202.0620
- Kieburg: arXiv:1202.1768

Three papers are still in preparation.