Invarian $\beta$-ensembles and the Gauss-Wigner crossover

Romain Allez, Jean-Philippe Bouchaud and Alice Guionnet
University Paris Dauphine, CFM and ENS Lyon-MIT

Gaussian Orthogonal/Unitary Ensembles

A random symmetric (resp. hermitian) matrix $H$ of size $N \times N$ belongs to the GOE (resp. GUE) if

$$P(dH) = \frac{1}{Z_N} \exp\left(-\frac{1}{2} tr(H^2)\right) dH.$$ 

For any $O \in O(N)$ (resp. $U(N)$),

$$OH \psi = H \psi$$

Hence $H$ have Haar distributed eigenvalues.

So the eigenvalues are independent of the eigenvectors and distributed according to the joint distribution with density (Dyson, 1962)

$$P_\beta(\lambda_1, \cdots, \lambda_N) = \frac{1}{Z_N} \prod_{i<j} |\lambda_i - \lambda_j|^\beta \exp\left(-\frac{1}{2} \sum_{i=1}^N \lambda_i^2 \right)$$

with $\beta = 1$ in the real symmetric case (resp. $\beta = 2$ in the complex hermitian case).

Dyson Brownian motion

Consider a sym. (resp. herm.) $M(t)$ s. t. for $i \leq j$,

$$dM_{ij}(t) = -\frac{1}{2} M_{ij}(t) dt + dB_{ij}(t)$$

where $B_{ij}(t)$ is a real Brownian motion and for $i < j$, $B_{ij}(t) = \sqrt{\frac{1}{2}} (B_{ij}^N(t) + \sqrt{\beta} - 1) B_{ji}^N(t)$, with $B_{ij}^N, B_{ji}^N$ indep. BMs. Nice thing is $t \rightarrow +\infty$:

$$M(t) \overset{law}{\rightarrow} H \in GOE \text{ (resp. GUE).}$$

Using perturbation theory,

$$d\lambda_i(t) = -\frac{1}{2} \lambda_i(t) dt + dB_i(t) + \frac{\beta}{2} \sum_{k \neq i} \lambda_k(t) dt$$

where the $b_i$ are independent BMs. For $\beta \geq 1$ (here in fact $\beta = 1$ or 2), there are almost surely no collisions between the particles.

Fokker Planck Eq. for the transition density $P(\lambda_1, \cdots, \lambda_N; t)$:

$$\frac{\partial P}{\partial t} = \sum_{i=1}^N \frac{\partial}{\partial \lambda_i} \left( \frac{1}{2} \lambda_i + \frac{\beta}{2} \sum_{k \neq i} \lambda_k - \lambda_i \right) P + \frac{1}{2} \sum_{i=1}^N \frac{\partial^2 P}{\partial \lambda_i^2}.$$ 

The stationary solution ($\frac{\partial P}{\partial t} = 0$) is indeed $P_\beta$ of Eq. (1).

Interpolation process

For $\beta = 1$ and $\beta = 2$, GOE and GUE are rotationally invariant matrix models with eigenvalues distributed according to the joint pdf $P_\beta$.

Purpose: construct a rotationally invariant matrix model with eigenvalues distributed as $P_\beta$ for all $0 < \beta < 2$.

Dumitriu & Edelman tridiagonal ensembles

Nice explicit matrix model

$$H^\beta = \frac{1}{\sqrt{2}} \begin{pmatrix} g_1 & \chi_{(N-1)\beta} & \cdots & \chi_{(2)\beta} & g_N \chi_{(1)\beta} \\ \chi_{(N-1)\beta} & g_2 & \cdots & \chi_{(3)\beta} & \cdots & \chi_{(2)\beta} & ... & \chi_{(1)\beta} \\ \vdots & \vdots & \ddots \end{pmatrix}$$

where the $g_i$ are gaussian variables and the $\chi_k$ are independent chi random variables indexed by their shape parameter.

Eigenvalues distributed according to $P_\beta$ for any $\beta > 0$ but obviously not rotationally invariant !

Density of eigenvalues

- If $p > 0$ does not depend on $N$, then the large $N$ stationary density is

$$\rho_N(\lambda) = \frac{1}{\sqrt{2\pi N}} e^{-\frac{1}{2} \lambda^2}.$$ 

- Wigner semicircle density.

- If $p = 0$, the particles evolve as independent BM, thus

$$\rho_N(\lambda) = \frac{1}{\sqrt{2\pi N}} e^{-\frac{1}{2} \lambda^2}.$$ 

Using Itô’s formula:

$$\frac{\partial G}{\partial t} = \frac{p N^2 G^2}{2 \frac{\partial x}{\partial z}} + \frac{\partial^2 G}{\partial x^2} - \frac{2-p}{2} \frac{\partial G}{\partial x}.$$ 

Further results: regime $p = 2c/N$

For $p = 2c/N$, the stationary solution satisfies

$$e_{G_N}(z) + e_{G_N}(z) + G_N(z) + 1 = 0.$$ 

With Riccati transform $G(z) = U(z)/cn(z)$, this can be solved ! This leads to

$$\rho_\beta(\lambda) = \frac{1}{\sqrt{2\pi (1+c)\lambda^2}}.$$ 

where

$$D_{\beta}(z) = \frac{e^{-z^2/4}}{\Gamma(c/2)} \int_0^{\infty} dx e^{-x^2} x^{c-1}.$$