Asymptotics for Toeplitz determinants: perturbation of symbols with a gap

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Toeplitz Matrices

We consider Toeplitz determinants of the form
\[ D_n(f) = \det(f_{j-k})_{j,k=0}^{n-1}, \quad f_j = \frac{1}{2\pi i} \int_{i\theta_0}^{i\theta_0} f(e^{i\theta}) e^{-i\theta j} d\theta, \]
where the symbol \( f \) is of the form
\[ f(e^{i\theta}) = e^{W(e^{i\theta})} \times \begin{cases} 1, & \text{for } -\theta_0 \leq \theta \leq \theta_0, \\ s, & \text{for } \theta_0 < \theta < 2\pi - \theta_0, \end{cases} \]
where \( 0 < \theta_0 < \pi, \) and \( W \) is analytic on \( S_1. \)

Three known results

- If \( s = 1, \) the weight \( f(e^{i\theta}) = e^{W(e^{i\theta})} \) is analytic.
  To get asymptotics, we can use the Szego strong limit theorem:
  \[ \ln D_n(s, \theta_0, W) = nW_0 + \sum_{k=1}^{\infty} kW_k W_{-k} + o(1), \]
  as \( n \to +\infty, \) where \( W_k = \frac{1}{2\pi i} \int_{i\theta_0}^{i\theta_0} W(e^{i\theta}) e^{-ik\theta} d\theta. \)
- If \( s = 0, \) the symbol is supported on an arc:
  \[ f(e^{i\theta}) = e^{W(e^{i\theta})} \times \begin{cases} 1, & \text{for } -\theta_0 \leq \theta \leq \theta_0, \\ 0, & \text{for } \theta_0 < \theta < 2\pi - \theta_0, \end{cases} \]
  Asymptotics for \( D_n \) are given by (Widom, 1971)
  \[ \ln D_n(s = 0, \theta_0, W) = n^2 \ln \sin \frac{\theta_0}{2} + nW_0 - \frac{1}{4} \ln n + \sum_{k=1}^{\infty} kW_k W_{-k} - \frac{1}{4} \ln \cos \frac{\theta_0}{2} \left( \frac{1}{12} \ln 2 + 3\zeta'(1) - 1 \right) + o(1), \]
  as \( n \to +\infty. \) These asymptotics are proved only if \( f \) is positive on its support and symmetric in \( \theta. \)
- If \( s \) is fixed, namely independent of \( n, \) \( f \) has two jump discontinuities which are special cases of Fisher-Hartwig singularities. Asymptotics for \( D_n \) are given by (Deift, Its-Krasovsky 2010, Widom, Basor, ...)
  \[ \ln D_n(s, \theta_0, W) = nW_0 + \left( \ln s \right)^2 - \frac{1}{2s^2} \ln n + \ln(2\sin \theta_0) + \sum_{k=1}^{\infty} kW_k W_{-k} - \frac{1}{4} \ln \cos \frac{\theta_0}{2} \left( \frac{1}{12} \ln 2 + 3\zeta'(1) - 1 \right) + o(1), \]
  as \( n \to +\infty, \) where \( G \) is Barnes' \( G \)-function.

Critical transition

The asymptotic expansion is different for \( s \to 0 \) in Fisher-Hartwig asymptotics than for \( s = 0. \) This shows that there is critical transition when \( s \to 0, \) depending on the speed of convergence of \( s \) to 0.

Theorem [Charlier, Claeys 2014]

Let \( \theta_0 \in (0, \pi) \) and define
\[ x_c = -2 \ln \tan \frac{\theta_0}{2}. \]
As \( n \to \infty \) and simultaneously \( s \to 0 \) in such a way that \( 0 \leq s \leq e^{-x_c n}, \) we have
\[ \ln D_n(s, \theta_0, W) = \ln D_n(0, \theta_0, W) + o(1). \]
If \( e < \theta_0 < \pi - \epsilon, \) \( o(1) = O(n^{-1/2} s e^{x_c n}). \)

We also generalise the result to the case where \( \theta_0 \to \pi, \pi - \theta_0 > \frac{M}{2} \) with \( M \) sufficiently large. Here
\[ o(1) = O((\pi - \theta_0)^{1/2} n^{-1/2} s e^{x_c n}). \]

Applications

- The Circular Unitary Ensemble.
  Define the random variable \( X \) as the number of eigenvalues of a CUE matrix on the arc
  \[ \gamma = \{ e^{i\theta} : \theta \in \left[ \pi - \theta_1, \pi + \theta_1 \right] \}, \]
  where \( \theta_1 \in [0, \pi - \theta_0] \) is the unique solution of
  \[ 2 \int_{-\theta_0}^{\theta_1} \log \left( \frac{1 + e^{im\theta}}{1 - e^{im\theta}} \right) u_s(e^{i\theta}) d\theta = x; \]
  and the density is given by
  \[ u_s(e^{i\theta}) = \frac{1}{2\pi} \left( \frac{\cos \theta + \cos \theta_0}{\cos \theta - \cos \theta_0} \right) \]
  The equilibrium measure is "regular" on \( S_1 \setminus J_\gamma. \)

Differential identity

\[ \partial_s \ln D_n(s, \theta_0, W) = \frac{1}{2\pi i} \int_{i\theta_0}^{i\theta_0} \left[ Y^{-1}(z)Y(z) \right]_{12} f(z, s) dz, \]
where \( Y \) is the unique solution of a RHP for orthogonal polynomials on the unit circle.
- We want to obtain asymptotics for \( Y \) using the steepest descent analysis.
- This requires transformations on \( Y \) to get an "easy" RHP.
- The first transformation uses the equilibrium measure \( u(e^{i\theta}) d\theta. \)

Equilibrium measure

We introduce the new parameter \( x : s = e^{-x n} \)
- For \( x \geq x_c, \)
  \[ u_s(e^{i\theta}) = \frac{1}{2\pi i} \left( \frac{\cos \theta + 1}{\cos \theta - \cos \theta_0} \right) \]
  and its support is \( \gamma = \{ e^{i\theta} : -\theta_0 \leq \theta \leq \theta_0 \}. \)
- \( x < x_c, \) it is "regular" on \( S_1 \setminus \gamma. \) If \( x = x_c, \) it is "regular" on \( S_1 \setminus \gamma \cup \{-1\}, \) and it is "non-regular" on \(-1, 1\).

Steepest descent analysis:
- Normalisation at infinity.
- Opening of the lenses.
- Global Parametrix.
- Local parametrices near \( e^{i\theta_0}, e^{-i\theta_0}, \) with 4 branches using Bessel functions.
- Local parametrix near \(-1, 1\).
- Final transformation.

Integration of the differential identity:

We started integration from 0:
\[ \ln D_n(s, \theta_0, W) = \ln D_n(0, \theta_0, W) + \int_0^s \partial_s \ln D_n(s, \theta_0, W) ds \]
The integral can be bounded using \( Y \) asymptotics.

Further developments

The case \( s > e^{-x_c n} \) RH analysis can be done:

Equilibrium measure

- For \( 0 < x < x_c, \) the support of the equilibrium measure consists of two disjoint arcs: we have
  \[ J_x = \gamma \cup \{ e^{i\theta} : \theta \in \left[ \pi - \theta_1, \pi + \theta_1 \right] \}, \]
  where \( \theta_1 \in [0, \pi - \theta_0] \) is the unique solution of
  \[ 2 \int_{-\theta_0}^{\theta_1} \log \left( \frac{1 + e^{im\theta}}{1 - e^{im\theta}} \right) u_s(e^{i\theta}) d\theta = x; \]
  and the density is given by
  \[ u_s(e^{i\theta}) = \frac{1}{2\pi} \left( \frac{\cos \theta + \cos \theta_1}{\cos \theta - \cos \theta_0} \right) \]
  The equilibrium measure is "regular" on \( S_1 \setminus J_x. \)

- The global parametrix involves elliptic \( \theta \) functions.
- Local parametrices near \( e^{i\theta_0}, e^{-i\theta_0}, \) using Airy functions.
- We still need to integrate the differential identity.
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