Fractal Weyl laws - beyond ballistic chaotic decay

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Weyl law

Qm energy levels

cl energy shell

Weyl: \( \#(E_n < E) = A/h^f \)
Open system

Complex energies $E$: $\text{Im } E_n = -\frac{\Gamma_n}{2}$

Question: $(E_n: \text{Re } E_n < E, \ \Gamma_n < \Gamma)$
Exploit qm-cl correspondence?

round-trip operator $F$, $\dim F = M = 1/h$; opening operator $P = (M \times N)$,

Inject a particle
t → E: Stroboscopic scattering theory

round-trip operator $F$, $\dim F = M = 1/h$; opening operator $P = (M \times N)$,
internal space: projector $Q = 1 - PP^T$

injection a particle:
exit: $P^T F P$
$P^T F (QF)P$
$P^T F (QF)^2 P$
$P^T F (QF)^3 P$
$P^T F (QF)^4 P$...

$S(\varepsilon) = P^T (e^{-i\varepsilon} - FQ)^{-1} FP$

Resonances: $QFQ\psi = e^{-i\varepsilon}\psi$; $e^{-i\varepsilon} \equiv \lambda$; $\varepsilon = E - i\Gamma/2$
Challenge: quasi-deterministic decay

- Nominally diverging decay rates: $|\lambda| = \exp(\text{Im} \varepsilon) = 0$
- Resonance wave functions quasi-degenerate (defective eigensystem)

$$\lim_{\Gamma \to \infty} \frac{1}{1 + e^{\Gamma(t-t_0)}}$$
illustration: standard map/kicked rotator

(classical)

\[
x_{n+1} = x_n + p_n \quad \text{(mod 1)}
\]

\[
p_{n+1} = p_n + \frac{K}{2\pi} \sin(2\pi x_{n+1}) \quad \text{(mod 1)}
\]

(qm)

\[
F_{nm} = \frac{1}{\sqrt{iM}} \exp\left[\frac{i\pi}{M} (m-n)^2 - \frac{iMK}{2\pi} \cos 2\pi \frac{m}{M}\right]
\]

K=7.5, M=1280, N=256

Resonances wave functions

Escape zones
• Classically chaotic systems (with J Tworzydło):
  fractal Weyl law
  – Goal: reinstate phase space rules

• Mixed phase space (with M Kopp):
  ... fractal Weyl law ...
  – Goal: test character of chaotic component

• Refractive escape (with M Kopp; J Wiersig; J Keating & M Novaes):
  – quantum dots with tunneling
  – dielectric resonators with refractive escape
Classically chaotic systems

Resonance distribution

- Resonance distribution plots for different values of $M$:
  - $M=640$
  - $M=320$
  - $M=160$
  - $M=80$

- Graph showing the scaled $P$ against $|\lambda|$ for RMT.
  - Fractal Weyl law
  - Power law scaling

- Formula for $P(\lambda)$ scaling:
  - $P(\lambda) \sim |\lambda|^{-\gamma}$ for $\gamma = 0.7, 0.4, 0.1$
Try to count short-living states

A. identify short-lived deterministic dynamics in phase space

\[ QFQ \psi_n = 0 \quad (\lambda_n = 0, \quad \Gamma_n = \infty) \]

- Define \( P = P^T P = 1 - Q \)
- trivially: \( Q P = 0 \rightarrow N \) states on opening (\( P_o = P \))

- semiclass.: preimage: projector \( P_1 = P_1 P_1^T \)
- naïve Weyl: \( \text{dim} = \text{area}/\text{Planck} = M \cdot \text{area} \)

\[ \boxed{\text{problem: underestimates no. of states}} \]
reason: operator not self-adjoint, states nonorthog., highly degenerate
B. Cure degeneracy

\[ QFQ \psi_n^{(1)} = 0 \quad (\lambda_n = 0) : \text{consider} \quad QFQ \psi_n^{(t+1)} = \lambda_n \psi_n^{(t+1)} + \psi_n^{(t)} = \psi_n^{(t)} \]

- 2\(^{\text{nd}}\) preimage, projector \( P_2 = P_2 P_2^T \)
- 3\(^{\text{rd}}\) preimage, projector \( P_3 = P_3 P_3^T \)
- \( t^{\text{th}} \) preimage, projector \( P_t = P_t P_t^T \)
- semiclassical propagation:
  \[ (QFQ)^t P_t = 0, \quad P_t P_s = 0 \quad (t \neq s) \]

C. Requires: areas \( A \approx \exp(-\Lambda t) > 1 / M \quad \Rightarrow \quad t < \frac{1}{\Lambda} \ln(M) \equiv t_{Ehr} \)

Weyl: \( \sum \text{rank } P_t = M (1 - e^{-t_{Ehr} / t_{dwell}}) \)

\[ t < t_{Ehr} \]

D. Remaining states (long living): \[ Me^{-t_{Ehr} / t_{dwell}} \propto M^{1-1/\Lambda t_{dwell}} \]
What have we done? A semiclassical partial Schur decomposition!

\[ \mathcal{P}_t : \text{part of orthogonal basis } U \text{ in } QFQ = UTU^+ \]

where \( T \) is triangular with evals on diagonal.

Test: Husimi rep. of Schur vectors (\( |\lambda_n| < 0.1, M=1280 \))

Husimi-Schur representation:

a new phase space representation for resonance eigenfunctions
Mixed phase space

\[ k = 2.0 \]

'fast' Husimi-Schur representation

'slow' Husimi-Schur representation
Mixed phase space

Position of leads is important; coupled islands: fast decay
Uncoupled islands: slow tunneling escape
Two accumulation regions:

\[ |\lambda| \approx 0.1 \]

- uncoupled islands (long-living states): just the ordinary Weyl law...
- idea: fix both upper and lower cut-off of lifetimes
typical lifetimes \((0.1 < |\mu_n| < 0.98)\)

opening at \(0 < q < 0.2\)
typical lifetimes ($0.1 < |\mu_n| < 0.98$)

opening at $0.2 < q < 0.4$. 
Slightly unexpected...

Time domain studies: classical part of mixed phase space is quite unlike a fully chaotic phase space:

Power law decay $\propto t^{-\alpha}$ instead of exponential decay $\propto \exp(-t/t_{dwell})$

Origin: sticking to islands (see eg Cristadoro/Ketzmerick PRL 08)
Possible explanation

Areas also shrink algebraically:

algebraic loss of qm-classical correspondence $\propto t^{-\beta}$

power-law Ehrenfest time $\tau_E \propto M^{1/\beta}$

Weyl law from combination of two power laws: $\tau_E^{-\alpha} \propto M^{-\alpha/\beta}$

(in contrast to $\exp(-\tau_E/\tau_{dwell}) = \dot{M}^{-1/\lambda \tau_{dwell}}$)
nonballistic escape I: tunnel barriers

e.g.: quantum dot with barrier at lead

Scattering operator:

\[
S(\varepsilon) = R' + T' \left( e^{-i\varepsilon} - FR \right)^{-1} FT
\]

\[
R = \begin{pmatrix}
1_{1\ldots i} \\
r_{1\ldots j} \\ 1_{j\ldots M}
\end{pmatrix}
\]

Some qm-cl correspondence remains

\[r^2 = 0 \quad \text{and} \quad r^2 = 1/2\]
Short-living resonances: life times strongly affected

$r^2=0$  
$r^2=1/2$
Apparent fractal Weyl law only at small reflection probability

\[ r^2 = 0.0001 \]
Scaling: cut-off dependent fractal Weyl law even at larger \( r \)

- \( r^2 = 0 \)
- \( r^2 = 0.1 \)
Smallest eigenvalue

$r^2 = 0.0001$

-independent of $k$ and $M$:

Universal scaling (like sqrt)
nonballistic escape II: dielectric resonators

Stroboscopic scattering operator

\[ S(\omega) = -R + T \left( e^{-i\omega \tau} - FR \right)^{-1} FT \]

with

• frequency \( \omega \),
• traversal time \( \tau = n \pi A / v C \) (Sabine’s law),
• \( R, T \) determined by Fresnel reflection coefficients.
(n: refractive index; A: area, C: perimeter, v: velocity)
• \( M=N=\dim S= \omega C/v \pi \) (Weyl’s law applied to the boundary)
Compare realistic resonator to random matrix theory (RMT)

Bands of short-living states (origin: bouncing ball motion)
Requires to renormalize $M$ and $\tau$! Here done independent from fluctuations by using mean level spacing and decay rate of long-living states.
Summary

- Phase space rules can be resurrected by semiclassical Schur decomposition; links fractal Weyl law to Ehrenfest time

- Fractal Weyl law also exists in generic dynamical systems (mixed phase space)

- Non-ballistic decay (q-dots with tunneling, dielectric resonators)
References

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