GSE statistics without spin

joint work with

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Spectral statistics

Random matrix conjecture

Spectra of chaotic systems have universal statistics in agreement with random matrix theory. Ensemble depends on symmetries. In absence of other symmetries it depends only on the behaviour under time reversal.
Spectral statistics

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Time reversal invariance

Classical: $H$ symmetric w.r.t. $p \rightarrow -p$.

Quantum: $\hat{H}$ symmetric w.r.t. complex conjugation, e.g.,

$$\hat{H} = \frac{1}{2}m\hat{p}^2 + U(x)$$

in general: $\hat{H}$ must commute with anti-unitary operator $T$

$$T(a|\psi\rangle + b|\phi\rangle) = a^* T|\psi\rangle + b^* T|\phi\rangle,$$

$$\langle T\psi|T\phi\rangle = \langle \psi|\phi\rangle^*$$

together with $T^2 |\psi\rangle = c |\psi\rangle$

this implies $T^2 = \pm 1$. 
Time reversal invariance

Conventional time-reversal invariance:

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$T |\psi \rangle + b |\phi \rangle = a^* T |\psi \rangle + b^* T |\phi \rangle$,

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\[ \mathcal{T}(a |\psi\rangle + b |\phi\rangle) = a^* \mathcal{T} |\psi\rangle + b^* \mathcal{T} |\phi\rangle, \quad \langle \mathcal{T} \psi |\mathcal{T} \phi\rangle = \langle \psi |\phi\rangle^* \]

Together with \( \mathcal{T}^2 |\psi\rangle = c |\psi\rangle \) this implies \( \mathcal{T}^2 = \pm 1 \)
Random matrix ensembles

-in absence of geometrical symmetries

- no time-reversal invariance: Gaussian Unitary Ensemble

- time-reversal invariance with $T^2 = 1$: Gaussian Orthogonal Ensemble

- time-reversal invariance with $T^2 = -1$: Gaussian Symplectic Ensemble
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Spin systems

e.g.: spin system with spin-orbit coupling

$$H = \hat{p}^2 + U(r) + \hbar^2 \sum_{i=1}^{3} \sigma_i L_i$$

commutes with

$$T = i \sigma_2$$

$$K = \text{compl. conjug.}$$

GSE statistics!

Hamiltonian can be brought to quaternion real form with blocks

$$H_{nm} = \left( \begin{array}{cc} \alpha & \beta \\ -\beta^* & \alpha^* \end{array} \right) = a_0 1 + a_1 i \sigma_1 \quad a_2 i \sigma_2 \quad a_3 i \sigma_3$$
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$L = r \times p$
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($K = \text{compl. conjug.}$)
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Main message

GSE statistics can arise without spin.

example: a quantum graph

background: discrete geometrical symmetries
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Quantum graphs

networks of vertices connected by bonds (with lengths)

Schrödinger equation on each bond

$$-\hbar^2 \frac{d^2}{dx^2} \psi(x) = E \psi(x)$$

conditions at the vertices:

e.g. continuity

+ Neumann conditions (sum over $d\psi/dx$ of adjacent bonds is 0)

large well connected graphs display RMT spectral statistics

if Hamiltonian and vertex conditions symmetric w.r.t. complex conjugation: GOE
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Quantum graphs
here time-reversal invariance is broken by a complex phase factor: GUE
Quantum graphs

The following graph has a symmetry $T = PK$ ($P =$ switching to other copy, $K =$ complex conjugation) $\Rightarrow$ GOE
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($P =$ switching to other copy, $K =$ complex conjugation)

$\mathcal{T}^2 = 1 \implies \text{GOE}$
Quantum graphs

The following graph has the anti-unitary symmetry $T$ defined by:

$$T\psi(x) = \begin{cases} \psi^\ast(Px) & x \in \text{left half} \\ -\psi^\ast(Px) & x \in \text{right half} \end{cases}$$

$$T^2 = -1 \Rightarrow \text{GSE}$$
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General approach to symmetries
Symmetries

Spectral statistics in systems with (discrete) geometric symmetries

Example: reflection symmetry
two subspectra:
eigenfunctions even under reflection ⇒ GOE
eigenfunctions odd under reflection ⇒ GOE
subspectra uncorrelated
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Spectral statistics in systems with (discrete) **geometric symmetries**?

**Example: reflection symmetry**

![Diagram of reflection symmetry with two subspectra](image)

Two subspectra:
- Eigenfunctions even under reflection $\Rightarrow$ GOE
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General discrete symmetries

- group of **classical** symmetry operations $g$
General discrete symmetries

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  in our example identity and reflection
General discrete symmetries

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- **quantum** symmetries
  
  $$U(g)\psi(r) = \psi(g^{-1}r)$$

  commute with Hamiltonian,
General discrete symmetries

- group of **classical** symmetry operations $g$
  - in our example identity and reflection

- **quantum** symmetries

  \[ U(g)\psi(r) = \psi(g^{-1}r) \]

  commute with Hamiltonian,

  they form a representation of the classical symmetry group, i.e.,

  \[ U(gg') = U(g)U(g') \]
General discrete symmetries

General discrete symmetries can diagonalize $H$ and block-diagonalize symmetry operators $U(g) = \begin{pmatrix} M^T_1(g) & \cdots & \cdots & \cdots & M^T_1(g) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ M^T_2(g) & \cdots & \cdots & \cdots & M^T_2(g) \end{pmatrix}$ blocks $M^\alpha(g)$ are (irreducible) matrix representations of the classical group, they satisfy $M^\alpha(gg') = M^\alpha(g)M^\alpha(g')$. Eigenfunctions corresponding to each block have same energy if they are grouped into a vector $\psi'\psi'\psi'$. We get:

$U(g)\psi'\psi'\psi' = M^\alpha(g)T\psi'\psi'\psi'$

Consider subspectra corresponding to irreducible representations
General discrete symmetries

- can diagonalize $H$ and **block-diagonalize** symmetry operators
General discrete symmetries

- can diagonalize $H$ and **block-diagonalize** symmetry operators

$$U(g) = \begin{pmatrix}
M_1^T(g) \\
\ldots \\
M_1^T(g) & M_2^T(g) \\
\ldots \\
M_2^T(g) & \ldots \\
\ldots
\end{pmatrix}$$

blocks $M_{\alpha}(g)$ are (irreducible) matrix representations of the classical group, they satisfy $M_{\alpha}(gg') = M_{\alpha}(g)M_{\alpha}(g')$. Each eigenfunction corresponding to each block has the same energy if grouped into a vector $\psi$, we get:

$$U(g)\psi = M_{\alpha}(g)\psi$$

consider subspectra corresponding to irreducible representations
General discrete symmetries

- can diagonalize $H$ and **block-diagonalize** symmetry operators

\[
U(g) = \begin{pmatrix}
M_1^T(g) & & \\
& \ddots & \\
& & M_1^T(g)
\end{pmatrix}
\begin{pmatrix}
& & \\
& \ddots & \\
& & \\
M_2^T(g) & & M_2^T(g)
\end{pmatrix}
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- blocks $M_\alpha(g)$ are (irreducible) matrix representations of the classical group, they satisfy
General discrete symmetries

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$$U(g)\psi = M_\alpha(g)^T\psi$$
General discrete symmetries

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types of representations:
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- complex $M_\alpha$
General discrete symmetries

types of representations:

- complex $M_\alpha$
- real $M_\alpha$
General discrete symmetries

types of representations:

- complex $M_\alpha$
- real $M_\alpha$
- quaternion real (pseudo-real) $M_\alpha$
Why? consider $T = \text{complex conjugation}; 2d \text{pseudo-real representation}$.

$\psi$ transform according to $U(g) \psi = M_\alpha(g) T \psi$ but $T \psi$ transforms with $(M_\alpha(g) T \psi) \ast \Rightarrow T$ not compatible with structure of subspace.

Use $\bar{T} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ instead: $\bar{T} \psi$ transforms as desired and $\bar{T}$ commutes with $H$.

$\bar{T}^2 = -1 \Rightarrow \text{GSE}$.
<table>
<thead>
<tr>
<th></th>
<th>no $T$ inv.</th>
<th>$T$ inv. ($T^2 = 1$)</th>
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<tbody>
<tr>
<td>complex rep.</td>
<td>GUE</td>
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Find a graph whose symmetry group has a pseudo-real representation.
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Find a graph whose symmetry group has a pseudo-real representation.
Construction of a GSE quantum graph

simplest group with a pseudo-real representation: quaternion group $Q_8 = \{\pm 1, \pm i, \pm j, \pm k : i^2 = j^2 = k^2 = ijk = -1\}$
elements can be written as products of the generators $I$ and $J$

Cayley graph: group elements as vertices
bonds of length connect group elements related by right multiplication with bonds of length connect group elements related by right multiplication with
Construction of a GSE quantum graph

simplest group with a pseudo-real representation:

\[
\begin{align*}
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\end{align*}
\]

Elements can be written as products of the generators \(i, j\), and \(k\). Cayley graph: group elements as vertices and bonds of length connect group elements related by right multiplication with bonds of length connect group elements related by right multiplication with
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![Cayley graph diagram]

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Construction of a GSE quantum graph
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- increase size:
Construction of a GSE quantum graph

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Graph with GSE subspectrum
Construction of a GSE quantum graph

... but boundary conditions mix pairs of degenerate eigenfunctions
Construction of a GSE quantum graph

- take fundamental domain (eighth of graph)
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Construction of a GSE quantum graph

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\[ I = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \]
Construction of a GSE quantum graph

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graph with pure GSE statistics
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Let each of the two eigenfunctions live on a separate copy of the graph with a pure GSE spectrum and no resemblance of spin.
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Numerical Results
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\[+1 + 1 - 1 - 1 + i + i - i - i\]
Numerical Results

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Further example

Spectral statistics in systems with (discrete) geometric symmetries?
Further example

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Example: symmetry w.r.t. rotations by $\frac{2\pi}{3}$
(Leyvraz, Schmit, Seligmann 96)
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$$\psi(r, \theta - \frac{2\pi}{3}) = \psi(r, \theta)$$
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\[ \psi(r, \theta - \frac{2\pi}{3}) = \psi(r, \theta) \quad \Rightarrow \text{GOE} \]
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\psi(r, \theta - \frac{2\pi}{3}) = e^{i\frac{2\pi}{3}} \psi(r, \theta)
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Example: symmetry w.r.t. rotations by $\frac{2\pi}{3}$
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![Diagram showing a shape with threefold rotational symmetry.]

eigenfunctions with

- $\psi(r, \theta - \frac{2\pi}{3}) = \psi(r, \theta)$ \implies \text{GOE}
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- $\psi(r, \theta - \frac{2\pi}{3}) = e^{i4\pi/3}\psi(r, \theta)$
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Spectral statistics in systems with (discrete) **geometric symmetries**?

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\psi(r, \theta - \frac{2\pi}{3}) &= \psi(r, \theta) \quad \Rightarrow \text{GOE} \\
\psi(r, \theta - \frac{2\pi}{3}) &= e^{i2\pi/3} \psi(r, \theta) \Rightarrow \text{GUE} \\
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