1 - $\mathcal{O}(n)$ model

For $n \in \mathbb{N}$, it admits a representation as a formal hermitian matrix model [1, 6]:

$$
\mathcal{W}_N(M, A_1, \ldots, A_n) = \frac{\text{Det} M}{\exp \left( \frac{N}{2} \sum_{i=1}^{n} A_i \right)}
$$

Sums over connected genus $g$ discrete surface with $k$ boundaries, built with:

- $n$ vertices
- $m$-gons ($m \geq 3$ if $d$ is fixed but arbitrary)
- $d$ triangles carrying a piece of path forming exactly $k$ loops
- a marked $g$-gon ($g \geq 1$) with a marked edge on an $i$-th boundary ($1 \leq i \leq k$)

We proved an algorithm to compute all $\mathcal{W}_N(M)$

2 - interest for the $\mathcal{O}(n)$ model

Critical points:
- Exhibits critical points [2] different from pure gravity at $t_c > 0$, for $0 < |n| \leq 2$
- For $n = -2$, 2-spin (g ∈ [2, 3]), several continuum limits:
  - Believed to be CFTs at gravity with:
    - $\epsilon = 1 - 6 \left( \frac{1}{\sqrt{g}} - 1/\sqrt{2} \right)$, where $g = (1 - \sqrt{g}) + 2p + 1$
  - Reach non-rational CFTs by the continuum limit of a microscopic model

Continuous:
- Counting discrete surfaces with additional structure
- Duality to $q = m^2$ Petts model
- Fully packed loops ↔ dimer configurations when $|v(x)| = x^2/2$

Matrix models:
- A direction of generalization of the algebraic geometry tools developed for the 1-matrix model ($n=0$) [3]

3 - The method of loop equations

Loop equations:
- Power law to prove automatically combinatorial recursion relations [7]

Lemma 0:
- When $x \to \infty$:

$$
\mathcal{W}_0(s) = \sqrt{i \pi} / \sqrt{x} \quad \text{or} \quad \mathcal{W}_0(s, f) \in \mathcal{O}(1/s^2)
$$

Combinatorial testing:
- In each variable (for $k \geq 1$), $\mathcal{W}_k(x, y)$ is homomorphic with one cut $\{a_j\}, \mathcal{B}(t) \subseteq C$

$$
\mathcal{W}_k(x, y) - \mathcal{W}_k(x, a_1) \propto \chi(x) - \chi(x - a_1)
$$

There exists a set of loop equations determining uniquely $\mathcal{W}_k$ satisfying these algebraic properties

4 - Analytical properties of $\mathcal{W}_k(x)$

The one-cut property implies $\forall \text{c} \in \{x_i, \{a_j\}\}$ and $x \to \infty$:

$$
\left[3\right] \mathcal{W}_3(x + i\epsilon) + \mathcal{W}_3(x - i\epsilon) + \mathcal{W}_3(-x) - \mathcal{V}(x)
$$

$$
\left[3\right] \mathcal{W}_3(x_i, x_j) + \mathcal{W}_3(x_i, x_j, x_k) - \mathcal{W}_3(x_j, x_k) + \mathcal{W}_3(x_i, x_k) - \mathcal{W}_3(x_i, x_k)
$$

$$
\left[4\right] \mathcal{W}_4(x_i, x_j, x_k) + \mathcal{W}_4(x_i, x_j, x_k, x_l) - \mathcal{W}_4(x_j, x_k, x_l) - \mathcal{W}_4(x_i, x_k, x_l) - \mathcal{W}_4(x_i, x_k, x_l) - \mathcal{W}_4(x_i, x_k, x_l)
$$

Riemann-Hilbert problem: solution in terms of functions on a spectral curve $\Sigma$

$$
\Sigma \in \{x \in C \quad C \subseteq \mathbb{C} \times \mathbb{R}\}
$$

5 - Topological recursion

> In [6], we have extended the topological recursion of [5] to the $\mathcal{O}(n)$ model
> One recovers the formalism valid for the 1-hermitian matrix for $n = 0$
> Once the spectral curve is found, few modifications arise when $n \neq 0$

**Physical sheet**
- We choose a maximal open set $\Sigma_0 \subseteq \Sigma$ such that
- a maps bijectively $\Sigma_0$ to $C \cup \{a\} \cup \{4\}$

**Branch points**
- We choose a set of simple zeroes of $\mathcal{W}_k(x) \cup \Sigma_2$

**Local involution**
- We define $\Omega$ defined locally around $a_k$ by $\mathcal{W}_k(x) = -x\mathcal{W}_k(x)$

**Bergman kernel**
- Unique differential form on $\Sigma$
- with prescribed double poles without residues at $x(a_0) = \pm x(a_2)$

**Recursion kernel**
- Unique solution of $\supset$ locally around $a_k$

**Correlation forms**
- For $2g - 2 + k > 0$, $\mathcal{W}_g^{(2)}(x_i, J)$ can be reached
- with a string of $2g - 2 + k$ residues at branch points.

**Free energies**
- For $g \geq 2, \mathcal{W}_g^{(2)}(x_i, J)$ can be reached with a string of $2g - 2$ residues.

> Let $\epsilon$ be a local primitive of $xy$ around $a_k$.

**The topological recursion generates with (angular) limits of spectral curves $\Sigma_k$**

$$
\mathcal{W}_k(x) = \text{const} \quad \text{or} \quad \mathcal{V}(x)
$$

6 - Study of critical points

> The most important one is the string susceptibility:

$$
\frac{d^2}{d \epsilon^2} \mathcal{W}_k^{(2)}(x_i, J) = 0
$$

> Planar $(g = 0)$ critical exponent and amplitudes were already known for small $k$

> KPZ scaling expected from Lioville th. predicts

$$
(2 - 2g - 4) \text{dependence of the general exponents}
$$

Pure gravity:
- No macroscopic loops in the continuum limit already exists in the 1-hermitian matrix model
- Reached by blowing up $\xi$ around a branch point which becomes a zero order of $a$ at $x = t_c$, $\mu = \frac{1}{2}$, $\gamma_{\text{int}} = -1/(1 + \mu)$

Dense phase (critical pt)
- Macroscopic loops filling densely the surface

$$
\text{Dilated phases (critical pt)}
$$
- Macroscopic loops and regions dominated by gravity simultaneously
- Reached when $\gamma$ is properly tuned while $x \to 0$, by blowing up $x \to x^{1/\gamma}$

$$
\mu = 0, \quad \gamma_{\text{int}} = -1 - \frac{1}{\gamma}
$$

Critical spectral curve

$$
\left[\mathcal{W}_k^{(2)}(x, y) = \mathcal{V}(x) / \gamma_{\text{int}} \quad \gamma_{\text{int}} = \frac{2^k + 2}{2^k + 1}
$$

**Proofs**

7 - Conclusion

> A microscopic model which has a non algebraic spectral curve is shown to satisfy the topological recursion

> Perspective: computing all possible boundary conditions

> Generalization to $\mathcal{O}(n)$ ensembles coupled to an $\mathcal{O}(n)$ model

Bibliography


