Eigenvalue statistics in non-Hermitean Wishart Random Matrices at $\beta = 2^*$

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Asymptotic analysis of the spectrum

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(outline)

1. Introduction to non-Hermitean Wishart Random Matrices and their applications.
2. Non-Hermitean Wishart with complex entries ($\beta = 2$).
3. Calculation of the joint probability density function for all eigenvalues and the $p$-point correlation function.
4. Asymptotic analysis of the density of states; various scaling regimes.
   - Interdisciplinary applications (Econophysics, bio-medical etc).
   - Physical applications (QCD).
5. Conclusions and open problems.
**Introduction to Wishart matrices**

**Classical Wishart matrices:**

1. Introduced by J. Wishart in 1928 in the context of multivariate statistics.

2. Definition: \( W = XX^\dagger \), here \( X \) is a rectangular \((n \times p)\) matrix with no specific symmetries.

3. Constructed from \( n \) sets of uncorrelated, discretized in time, Gaussian random processes \( x_\alpha(t_j) \) as

\[
X_{\alpha,j} = x_\alpha(t_j), \quad W_{\alpha,\beta} = \sum_{j=1}^{p} X_{\alpha,j} \bar{X}_{\beta,j}, \quad \alpha, \beta = 1, \ldots n
\]

\( W \) is the most random correlation matrix.
Applications

1. **Wishart matrix** (or most random correlation matrix) acts like a template or as a null hypothesis in testing correlations within a given complex system.

2. The **empirical correlation matrix** constructed from the experiment (data) contains the information about correlations between time-series. Consider an example from Econophysics. Let \( x_\alpha(t) \) represents the time-dependent price changes of the \( \alpha \)-th asset (say gold items belonging to a person). The empirical correlation matrix between \( \alpha \)-th and \( \beta \)-th assets is defined as:

\[
C_{\alpha,\beta} = \sum_{j=1}^{p} x_\alpha(t_j) x_\beta(t_j), \quad \alpha, \beta \in (1, \ldots, n)
\]

3. If \( x_\alpha(t_j) \) and \( x_\beta(t_j) \) are uncorrelated Gaussian random processes, then \( C_{\alpha,\beta} \rightarrow W_{\alpha,\beta} \)

4. In the case of correlations, the maximal eigenvalue of \( C \) is much bigger than that of the Wishart matrix (of same dimensions and variance chosen for the best fit\(^1\)). It is then expected that the eigenvector corresponding to this large eigenvalue of \( C \) contains important correlation information.

\(^1\)See for example; L. Laloux et al, PRL, 83, 1467 (1999)
Spectral properties of the Wishart random matrices

1. Wishart matrix \( W_{\alpha,\beta} = \sum_{j=1}^{p} X_{\alpha,j} \bar{X}_{\beta,j}, \) \( \alpha, \beta = 1, \ldots, n \) \((p > n)\).

2. The J. P. D. F of its eigenvalues:
\[
P_n^{(\beta)}(\lambda_1, \ldots, \lambda_p) \propto \prod_{j>k=1}^{p} |\lambda_j - \lambda_k|^\beta \prod_{j=1}^{p} \lambda_j^{\frac{\beta}{2}(n-p+1)-1} e^{-\lambda_j/2}
\]

3. The mean density of eigenvalues in the limit \( n = N \to \infty \) with \( p/N = Q \) kept fixed is \( \rho W(\lambda) \), with \( Q = p/N \) is the "aspect ratio", and
\[
\lambda_{max/min} = 1 + 1/Q \pm 2/\sqrt{Q}.
\]

4. Characteristic eigenvalue gap (no \( \lambda \) for \( \lambda < \lambda_{min} \)), and upper-edge cut (no \( \lambda \) for \( \lambda > \lambda_{max} \)) for infinite sized matrices.

5. Hermitean anti-Wishart regime \((p < n, \ Q \to 1/Q)\) does not bring any new qualitative results.

\[\begin{align*}
\rho W(\lambda) &= \frac{Q}{2\pi} \frac{\sqrt{(\lambda_{max} - \lambda)(\lambda - \lambda_{min})}}{\lambda} \\
\end{align*}\]

[Marcenko-Pastur law]
In problems where time-series generating systems are two different systems and the interest is in studying the correlations between them (e.g. correlations between functioning of the left and right hemispheres of a brain) the correlation matrix takes the form $\tilde{W} = XY^\dagger$. We call this the non-Hermitean Wishart random matrix;

1. $W_{\alpha,\beta} = \sum_{j=1}^{p} X_{\alpha,j} \bar{Y}_{\beta,j}, \quad \alpha, \beta = 1, \ldots, n, \quad (p > n)$. The matrix is no longer Hermitean and its spectrum becomes complex valued.

2. What are the differences between well studied Ginibre and non-Hermitean Wishart?

3. In the following we consider non-Hermitean Wishart matrices with complex entries ($\beta = 2$).


$^2$ J. Kwapien et al., PRE, 62, 5557 (2000)
Analytic theory for non-Hermitean Wishart random matrices

Starting point: J.P.D.F. of matrix elements:

\[ P_{n,p}(W) DW = \langle \delta(2n^2)(W - XY^\dagger) \rangle_{X,Y} DW = \int \frac{1}{\text{Det}^p(1+Y_nY_n^\dagger)} \text{Exp}[i\text{tr}(W^\dagger Y + W Y^\dagger)]DY_n DW. \]

Goal: J. P. D. F. of all eigenvalues and correlation functions.

1. First Schur-decompose \( W \) as, \( W = UR_nU^\dagger \)
2. The Jacobian of transformation is:
   \[ DW = |\Delta_n(W)|^2 (dU) \prod_{j=1}^n dw_j D\tilde{R}_n. \]
3. Using the cyclic property of the trace in the exponent and delta functions from the integrals over \( R_n \), we conclude that \( X_n \) is lower triangular matrix, and,
4. \[ P_{n,p}(W) DW = C_U \int |\Delta_N(w)|^2 e^{i(w_jx_j^\dagger + \bar{w}_jx_j)} \prod_{j=1}^N dw_j \prod_{j=1}^n d^2 x_{jj} \]
   \[ \times \int \frac{D\tilde{X}_n}{\text{det}^p(\mathbb{I} + X_nX_n^\dagger)}. \]
We solve by using successive size reduction of the matrix $X_n$, decompose $X_n$ as:

1. \[ \mathbb{I} + X_n X_n^\dagger = \mathbb{I} + \begin{pmatrix} X_{n-1} & \phi \\ u^\dagger & x_{nn} \end{pmatrix} \begin{pmatrix} X_{n-1}^\dagger & u \\ \phi & x_{nn}^* \end{pmatrix} \]

2. By decomposing $D\tilde{X}_n = (du \ du^\dagger)D\tilde{X}_{n-1}$, we derive,

\[
P_{n,p}(W)DW = C_U \int |\Delta_n(w)|^2 e^{i(w_j x_{jj}^\dagger + \bar{w}_j x_{jj})} \prod_{j=1}^n dw_j \prod_{j=1}^n d^2x_{jj} \int \frac{D\tilde{X}_{n-1}}{\det^p(\mathbb{I} + X_{n-1} X_{n-1}^\dagger)^{-1} X_{n-1} u)} du \ du^\dagger
\]

\[
\times \int \frac{du \ du^\dagger}{(\mathbb{I} + u^\dagger u + |x_{nn}|^2 - u^\dagger X_{n-1}^\dagger(\mathbb{I} + X_{n-1} X_{n-1}^\dagger)^{-1} X_{n-1})^p} \prod_{j=1}^n d^2x_{jj}
\]

3. With a change of co-ordinates as

\[
v = (1 + |x_{nn}|^2)^{-1/2}(\mathbb{I} - X_{n-1}^\dagger(\mathbb{I} + X_{n-1} X_{n-1}^\dagger)^{-1} X_{n-1})^{1/2} u\]

and with some manipulations, $P_{n,p}(W)DW$ reduces to,

\[
C'_U \int |\Delta_n(w)|^2 e^{i(w_j x_{jj}^\dagger + \bar{w}_j x_{jj})} \prod_{j=1}^n dw_j \frac{\prod_{j=1}^n d^2x_{jj}}{(1 + |x_{nn}|^2)^{p-n+1}} \int \frac{D\tilde{X}_{n-1}}{\det^{p-1}(\mathbb{I} + X_{n-1} X_{n-1}^\dagger)}
\]

4. Then $p$ recursions reduce it to a single integral, whose exact solution finally gives us:
The joint probability density function of all eigenvalues and the density of states

1. \[ P(w_1, \ldots, w_n) = C \prod_{j=1}^{n} |w_j|^\nu K_\nu(2|w_j|)|\Delta_n(w)|^2, \text{ and } \nu = p - n. \]

2. \[ C = \left(\frac{\pi}{2}\right)^{-n} (\prod_{j=1}^{n} [\Gamma(j)\Gamma(j + \nu)])^{-1} \]

3. The Dyson integration theorem brings the density of states after successive integrations as:

\[ \rho(|w|) = \frac{2}{\pi} |w| K_\nu(2|w|) \sum_{k=0}^{n-1} \frac{|w|^{2k}}{k!(k + \nu)!} F_{p,n}(|w|^2) \]

4. This is also obtained by, J. C. Osborn, PRL, 93 222001 (2004), for \( \beta = 1 \) case see: G. Akemann, M.J. Phillips, H.-J. Sommers. They use a different method called "variational method". But the asymptotic analysis is new here.
Asymptotic analysis:

Case I: Interdisciplinary applications (Econophysics, bio-medical etc)

Solution by constructing a differential equation and a scaling ansatz

1. Define: \( F_{p,n}(x) = \sum_{k=0}^{n-1} \frac{x^k}{k!(k+\nu)!} \).

2. By differentiating one can construct the differential equation:
\[
xF''_{p,n}(x) + (\nu + 1)F'_p(x) - F_p(x) = -\frac{x^{n-1}}{\Gamma(n)\Gamma(n+\nu)}
\]

3. Scaling ansatz: \( F_{p,n}|_{x=t\nu^c} = g(\nu, n)f_n(t), \quad \nu \to \infty \).

The solution of the above differential equation gives us: \( F_{p,n}(x) = e^{\frac{x}{\nu}} \frac{\Gamma(n, \frac{x}{\nu})}{\nu \Gamma(n)} \),

and the mean density takes the compact form,

\[
\rho_{p,n}(|w|) \simeq \frac{1}{\pi \nu} \frac{\Gamma(n, \frac{|w|^2}{\nu})}{\Gamma(n)}
\]

4. For \( n = N >> 1 \), we have the Girko's law: \( \frac{1}{\pi \nu} \Theta(\sqrt{\nu N} - |w|) \).

5. Alternatively, the same formula can be obtained from the direct analysis of the series \( \sum_{k=0}^{n-1} \frac{x^k}{k!(k+\nu)!} \) in the large \( \nu \) limit.
Asymptotic analysis:

Case II: Physical problems regime (QCD motivated case, large $N$ but finite $\nu$)

1. For $N \to \infty$ the sum $\sum_{k=0}^{N-1} \frac{x^k}{k!(k+\nu)!} \to |w|^{-\nu} I_{\nu}(2|w|)$. The mean density in the large $N$ limit is: $\rho_{p,N}(|w|) = \frac{2}{\pi} K_{\nu}(2|w|)I_{\nu}(2|w|)$.

2. Behaviour near origin ($|w| \to 0$):

$$\rho_{p,N}(|w|) = \begin{cases} -\frac{2}{\pi} \log(|w|) & \text{for } \nu = 0 \\ \frac{1}{\pi \nu} = \text{Const.} & \text{for } \nu \neq 0 \end{cases}$$

3. Behaviour at large $w$, $1 << w << N$: $\rho_{p,n}(|w|)|_{N \to \infty} = \frac{1}{2\pi |w|}$. This is in contrast with the constant density profile in the Ginibre case: $\rho^{\beta=2}_{Ginibre}(x\sqrt{N}) = \frac{1}{\pi}$ for $x < 1$ and zero otherwise.
Breakdown of $\frac{1}{2\pi|w|}$ – law at $|w| = |w_c|$

$$2\pi \int_0^{|w_c|} |w| \left(\frac{1}{2\pi|w|}\right) d|w| = N, \quad \Rightarrow |w_c| = N.$$ Thus, something non-trivial should happen at about $|w| \sim N$. To understand it we have to treat the large $N$ limit more carefully.

Idea is to utilize the Euler-Maclaurin formula to reduce the sum

$$F_{p,N} = \sum_{k=0}^{N-1} \frac{x^k}{k!(k+\nu)!}$$

into an integral in the large $N$ limit, and solve it using the saddle point approximation.

Case I: $|w| < N$

$$\rho_{p,N}(|w|) = \frac{1}{4\pi|w|} \left(\text{Erf}(\sqrt{|w|}) - \text{Erf}\left(\frac{N - |w|}{\sqrt{|w|}}\right)\right)$$
**Tails of the density (rare fluctuations)**

1. **Case II:** $|w| > N$

\[
\rho_{p,N}(|w|) = \frac{1}{4\pi|w|} \left(1 + \text{Erf}\left(\frac{N - |w|}{\sqrt{|w|}}\right)\right)
\]

2. **Number of eigenvalues in the tails (i.e., for $|w| > N$)**

\[
N_+ \approx \frac{1}{2} \sqrt{\frac{N}{\pi}} + \frac{1}{8}
\]
Conclusions

- An exact treatment of non-Hermitean Wishart random matrices at $\beta = 2$.
- Various large $-N$ / large-$\nu$ scaling regimes analyzed.
- Good agreement with numerical simulations observed.
Open problems

- The non-Hermitean Wishart at $\beta = 1$ symmetry class (work in progress). In this case spectrum is complex valued as in the case $\beta = 2$, but there is a finite probability of real eigenvalues (due to the accumulation of eigenvalues along the real axis).

- In applications where the number of discretizations ($p$) is less than number of channels ($N$) (for example, due to unavailability of data points), we are in the non-Hermetian anti-Wishart regime. This is a completely open area.

- Experimental utilization of the present work remains open, although in such cases people have relied on Ginibre ensembles [see for example, J. Kwapien et al, arXive:physics/0605115]. The present work will improve our understanding of such applications.