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TRENDS AND CYCLES IN MACRO SERIES: THE CASE OF US REAL GDP

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THE CASE OF US REAL GDP

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Abstract

In this paper we propose a new modelling framework for the analysis of macro series that includes both stochastic trends and stochastic cycles in addition to deterministic terms such as linear and non-linear trends. We examine four US macro series, namely annual and quarterly real GDP and GDP per capita. The results indicate that the behaviour of US GDP can be captured accurately by a model incorporating both stochastic trends and stochastic cycles that allows for some degree of persistence in the data. Both appear to be mean-reverting, although the stochastic trend is nonstationary whilst the cyclical component is stationary, with cycles repeating themselves every 6 – 10 years.

Keywords: GDP; GDP per capita; trends; cycles; long memory; fractional integration

JEL Classification: C22, E32

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1. Introduction

In this paper we put forward a new modelling framework for macro series that allows for two singularities (or poles) in the spectral density function, one corresponding to the long-run or zero frequency (i.e. to the long-run evolution of the series), the other to a non-zero frequency (and related to a cyclical pattern repeated approximately every 6 – 10 years). The proposed model is very general: it extends the classical framework based on (non-cyclical and cyclical) unit roots by allowing for fractional integration, and considers both deterministic and stochastic patterns, at the zero and cyclical frequencies, including both long- and short- memory components. The cyclical patterns are modelled using Gegenbauer processes.

The suggested methodology is then applied to analyse the stochastic behaviour of four US macro series, namely annual and quarterly GDP and GDP per capita. The results indicate that the behaviour of US GDP can be captured accurately by a model incorporating both stochastic trends and stochastic cycles that allows for some degree of persistence in the dynamics of the series. Both appear to be mean-reverting, although it is found that the stochastic trend is non-stationary whilst the cyclical component is stationary, with cycles repeating themselves every 6 – 10 years. Deterministic (linear and non-linear) terms were also incorporated into the model but were not found to be statistically significant in any case.

The layout of the paper is as follows. Section 2 briefly reviews the main approaches to modelling GDP found in the literature. Section 3 presents the statistical model. Section 4 discusses the data and the empirical results. Section 5 offers some concluding remarks.

2. Literature Review

GDP, whether nominal, real or per capita, is typically a non-stationary variable in most developed countries. For many years, the standard modelling approach was to use deterministic functions of time, usually of a linear form, as in the following specification:

$$y_t = \alpha + \beta t + x_t, \quad t = 1, 2, ..., \qquad (1)$$

where $\{y_t, t = 1, 2, \ldots, T\}$ is the observed (GDP) series, α and β are the coefficients on an intercept and a linear time trend respectively, and x_t is assumed to be covariance stationary, usually of the ARMA form, to capture short-run and cyclical patterns in the data. Therefore, the process followed by x_t can be represented as

$$\phi(L) x_t = \theta(L) \varepsilon_t, \quad t = 1, 2, ..., \qquad (2)$$

where $\phi(L)$ and $\theta(L)$ stand for the AR and MA components of the series respectively. This modelling framework was dominant in the literature until the publication of a very influential paper by Nelson and Plosser (1982), who examined fourteen US macroeconomic series and by applying the tests developed by Fuller (1976) and Dickey and Fuller (1979) found evidence of unit roots and came to the conclusions that the behaviour of these variables except one could be better described in terms of stochastic trends, that is, as in the following model including an intercept:

$$y_t = \alpha + y_{t-1} + x_t, \quad t = 1, 2, ..., \qquad (3)$$

where x_t is I(0) and can be represented as in (2).[1] This model has been widely employed in the macro literature and in the last twenty years many additional unit root tests have been developed (Phillips and Perron, 1988; Elliot et al., 1996; Ng and Perron, 2001; etc.). These two specifications, i.e. the deterministic trend model as in (1) and the

[1] For our purposes we define an I(0) process as a covariance stationary process, i.e. one for which the infinite sum of the autocovariances is finite. Alternatively, in the frequency domain, it can be defined as a process with a spectral density function that is positive and finite at all frequencies in the spectrum.

stochastic trend model as in (3), can coexist within the same framework if x_t in (1) contains a unit root, the main difference between the two models being the treatment of shocks, which have transitory effects in the case of (1) but permanent ones in the case of (3). However, a process may display nonstationary, persistent behaviour but still be mean-reverting as in the I(d) models with a differencing parameter d lying in the interval [0.5, 1). In such models, x_t is specified as

$$(1 - L)^d x_t = u_t, \quad t = 1, 2, ..., \qquad (4)$$

where d can be any real value and u_t is I(0) (defined as in footnote 1). Variants of this model have been used to analyse the behaviour of GDP in various countries (see, e.g., Michelacci and Zaffaroni, 2000, Mayoral, 2006, Gil-Alana, 2010, Caporale and Gil-Alana, 2013, Caporale and Skare, 2014).

Cyclicality is another important feature of GDP series. There exists a large literature using different methods such as time-varying transition probabilities (TVTP) Markov-switching regime models (see, e.g., Simpson et al., 2001), band pass filters (Christiano and Fitzgerald, 1999), etc. A similar approach to equation (4) can also be used to allow stochastic cyclical processes to be fractional as in the following model,

$$(1 - 2\cos\mu L + L^2)^d x_t = u_t, \qquad t = 1, 2, ..., \qquad (5)$$

with

$$(1 - 2\mu L + L^2)^{-d} = \sum_{j=0}^{\infty} C_{j,d}(\mu) L^j ,$$

where L stands for the lag operator (i.e., $Lx_t = x_{t-1}$), $C_{j,d}(\mu)$ are orthogonal Gegenbauer polynomial coefficients defined recursively as:

$$C_{0,d}(\mu) = 1, \qquad C_{1,d}(\mu) = 2\mu d,$$

$$C_{j,d}(\mu) = 2\mu\left(\frac{d-1}{j} + 1\right) C_{j-1,d}(\mu) - \left(2\frac{d-1}{j} + 1\right) C_{j-2,d}(\mu), \quad j = 2, 3, ...,$$

Gray et al. (1989, 1994) showed that x_t in (5) is (covariance) stationary if $d < 0.5$ for $|\mu = \cos w_r| < 1$ and if $d < 0.25$ for $|\mu| = 1$. This process implies the existence of a pole or singularity at a non-zero frequency which corresponds to the cyclical pattern. Special cases of this model were analysed by Athola and Tiao (1987) and Bierens (2001) setting d = 1, and by Gil-Alana (2001), DePenya and Gil-Alana (2006) and others allowing d to take fractional values. In this paper we combine the two models given by equations (4) and (5) in a single framework that is presented in the following section.

3. A New Statistical Model for Trends and Cycles

We propose a very general specification that incorporates both deterministic and stochastic trends, not only at the zero frequency but also at the non-zero (cyclical) frequencies, allowing for both long- and short memory components. The model is the following:

$$y_t = f(\varphi, t) + x_t, \quad t = 1, 2, \ldots,$$

$$(1 - L)^{d_1} (1 - 2\cos w_r L + L^2)^{d_2} x_t = u_t, \quad t = 1, 2, \ldots, \qquad (6)$$

$$\phi(L) u_t = \theta(L) \varepsilon_t, \quad t = 1, 2, \ldots,$$

where f is a function that can also be non-linear, depending on time and the unknown parameter vector φ; L is the lag operator; d_1 and d_2 are the integration orders of the long-run and the cyclical frequency respectively, where $w_r = 2\pi r/T$ with $r = T/j$, j indicates the number of periods per cycle and r the frequency with a singularity or pole in the spectrum; u_t is I(0) and displays weak dependence (as in equation (2) but replacing x_t with u_t), and ε_t is i.i.d. $N(0, \sigma^2)$. Therefore, the vector of parameters to be estimated is $\psi = [\varphi^T, d_1, d_2, r, \phi_1, \ldots, \phi_p, \theta_1, \ldots \theta_p]^T$.

Robinson (1994) had previously proposed a general testing framework that includes as a special case a very similar specification to ours. He considered the following model with a linear deterministic component:

$$y_t \;=\; \beta^T z_t \;+\; x_t\,, \quad t = 1, 2, ...,$$

$$(1 - L)^{d_1} \prod_{j=2}^{s} (1 - 2\cos w_r^j L \;+\; L^2)^{d_j} x_t \;=\; u_t\,, \quad t = 1, 2, ..., \qquad (7)$$

$$\phi(L) u_t \;=\; \theta(L) \varepsilon_t\,, \quad t \;=\; 1,\, 2,\, ...,$$

where z_t is a (kx1) vector of deterministic terms and/or weakly exogenous variables and s represents the number of cyclical and/or seasonal patterns observed in the data. Within this set-up, he tested the null hypothesis:

$$H_o : d \;=\; d_o, \qquad (8)$$

where $d = [d_1, d_2, \ldots, d_s]^T$ and $d_o = [d_{1o}, d_{2o}, \ldots, d_{so}]^T$ is a (s x 1) vector of given real numbers.

Assuming now that s = 2, the second equation in (7) becomes the same as the second one in (6), with the two coefficients d_1 and d_2 referring respectively to the long run and cyclical components of the series. In this context, a LM test of (8) in (6) can be defined as

$$\hat{R} \;=\; \frac{T}{\hat{\sigma}^4} \hat{a}' \hat{A}^{-1} \hat{a}, \qquad (9)$$

where T is the sample size, and

$$\hat{a} \;=\; \frac{-2\pi}{T} \sum_j^{*} \psi(\lambda_j) g_u(\lambda_j; \hat{\tau})^{-1} I(\lambda_j); \qquad \hat{\sigma}^2 \;=\; \sigma^2(\hat{\tau}) \;=\; \frac{2\pi}{T} \sum_{j=1}^{T-1} g_u(\lambda_j; \hat{\tau})^{-1} I(\lambda_j),$$

$$\hat{A} \;=\; \frac{2}{T} \left(\sum_j^{*} \psi(\lambda_j) \psi(\lambda_j)' \;-\; \sum_j^{*} \psi(\lambda_j) \hat{\varepsilon}(\lambda_j)' \left(\sum_j^{*} \hat{\varepsilon}(\lambda_j) \hat{\varepsilon}(\lambda_j)' \right)^{-1} \sum_j^{*} \hat{\varepsilon}(\lambda_j) \psi(\lambda_j)' \right)$$

$\psi(\lambda_j)' = [\psi_1(\lambda_j), \psi_2(\lambda_j)]$; $\hat{\varepsilon}(\lambda_j) = \frac{\partial}{\partial \tau} \log g_u(\lambda_j; \hat{\tau})$; $\psi_1(\lambda_j) = \log \left| 2 \sin \frac{\lambda_j}{2} \right|$;

$\psi_2(\lambda_j) = \log \left| 2(\cos\lambda_j - \cos w_r) \right|$, where $\lambda_j = 2\pi j/T$, and the summation in * in the above equations is over all frequencies which are bounded in the spectrum. $I(\lambda_j)$ is the periodogram of $\hat{u}_t$ defined as:

$$\hat{u}_t = (1-L)^{d_{1o}} (1 - 2\cos w_r L + L^2)^{d_{2o}} y_t - (1-L)^{d_{1o}} (1 - 2\cos w_r L + L^2)^{d_{2o}} f(\hat{\varphi}, t),$$

where the last term of the above equation, for some special non-linear cases such as those presented in the following section, can be expressed in a linear way as $\hat{\varphi}\ \bar{f}(t))$, with

$$\hat{\varphi} = \left(\sum_{t=1}^{T} \bar{f}(t)\ \bar{f}(t)' \right)^{-1} \sum_{t=1}^{T} \bar{f}(t) (1-L)^{d_{1o}} (1 - 2\cos w_r L + L^2)^{d_{2o}} y_t;$$

evaluated at $\lambda_j = 2\pi j/T$. Also, $\hat{\tau} = \arg\min_{\tau \in T^*} \sigma^2(\tau)$, with T^* as a suitable subset of the R^q Euclidean space. Finally, g_u is a known function coming from the spectral density of u_t:

$$f_u(\lambda) = \frac{\sigma^2}{2\pi} g_u(\lambda; \tau), \qquad -\pi < \lambda \leq \pi.$$

Note that these tests are parametric and, therefore, they require specific modelling assumptions about the short-memory specification of u_t. In particular, if u_t is a white noise, $g_u \equiv 1$, whilst if it is an AR process of the form $\phi(L)u_t = \varepsilon_t$, $g_u = |\phi(e^{i\lambda})|^{-2}$, with $\sigma^2 = V(\varepsilon_t)$, with the AR coefficients being a function of τ.

The point estimates were obtained by choosing over a grid the values of d_1, d_2 and r that minimise Robinson's (1994) test statistic. They were found to be almost the same as those obtained by maximising the Whittle function in the frequency domain (Dahlhaus, 1989). The confidence intervals were calculated by choosing the values of the differencing parameters for which the null hypothesis could not be rejected at the 5% level.

Under very general regularity conditions, Robinson (1994) showed that for this particular version of his tests:

$$\hat{R} \rightarrow_d \chi_2^2, \quad as \quad T \rightarrow \infty, \qquad (10)$$

where "$\rightarrow_d$" stands for convergence in distribution. Therefore, unlike in the case of other procedures, this is a classical large-sample testing situation. If this test is carried out in the context of (6) the null H_o will be rejected against the alternative H_a: $d \neq d_o$ if $\hat{R} > \chi^2_{2,\alpha}$, with Prob ($\chi_2^2 > \chi^2_{2,\alpha}$) = α. As mentioned before, despite the potentially non-linear structure of the first equation in (6), its interaction with the second equation will make it linear for some specific nonlinear structures, such as the Chebyshev polynomials in time presented in the following section.[2]

4. Empirical Analysis

We examine the following four series:

1) US annual real Gross Domestic Product,
2) US annual real Gross Domestic Product per capita,
3) US quarterly real Gross Domestic Product,
4) US quarterly real Gross Domestic Product per capita,

for the time period from 1929 – 2015 in the case of annual data, and from 1947Q1 till 2015Q3 in case of the quarterly data.

[Insert Figures 1 - 3 about here]

[2] A linear specification of this model combining stochastic trends and cycles for financial data can be found in Caporale and Gil-Alana (2017).

Figure 1 displays the four series, all of which exhibit an upward trend suggesting non-stationary behaviour. This is confirmed by their correlograms (Figure 2) and the periodograms (Figure 3), the former decaying slowly and the latter exhibiting their highest values at the smallest frequencies. Figure 4 displays the same four series in first differences, with the corresponding correlograms and periodograms (displayed in Figures 5 and 6 respectively) providing evidence of cyclical patterns.

[Insert Figures 4 - 6 about here]

We start by considering a linear model with a time trend allowing for unit roots and fractional degrees of integration, specifically:

$$y_t = \alpha + \beta t + x_t, \quad (1 - B)^d x_t = u_t, \quad t = 1, 2, ..., \tag{11}$$

where the errors are assumed to follow in turn a white noise and an autocorrelated process. However, instead of imposing a parametric ARMA structure on u_t, we employ a non-parametric method due to Bloomfield (1973) such that the error term is specified exclusively in terms of its spectral density function, which is given by

$$f_u(\lambda; \tau) = \frac{\sigma^2}{2\pi} \exp\left(2 \sum_{r=1}^{m} \tau_r \cos(\lambda r) \right), \tag{12}$$

where σ^2 is the variance of the error term and m indicates the number of short-run dynamic terms, usually low (e.g., 1 or 2), which approximates highly parameterised ARMA models with very few parameters, and producing autocorrelations decaying exponentially as in the AR case. Moreover, it is stationary for all range of parameters unlike in the AR case.

Tables 1 and 2 display the estimates of d, along with their corresponding 95% confidence bands, for the three cases of i) no deterministic terms, ii) a constant, and iii) a constant and a linear time trend, assuming in turn that u_t is a white noise (Table 1) and autocorrelated as in the model of Bloomfield (Table 2).

[Insert Tables 1 and 2 about here]

In the white noise case the time trend is significant in all cases except annual GDP per capita, and the estimates of d are significantly above 1, ranging from 1.31 (quarterly GDP) to 1.45 (annual GDP per capita). When allowing for (weak) autocorrelation as specified by Bloomfield (1973), the time trend is significant in all four cases, and the estimated values of d are still significantly above 1 but smaller.

Given the significance of the time trend in most cases, next we investigate whether it might be non-linear by using an approach based on Chebyshev polynomials in time that has been shown to perform well in the context of the tests of Robinson (1994) for fractional integration (Cuestas and Gil-Alana, 2016). Thus, we replace the first (linear) equation in (11) with:

$$y_t \;=\; \sum_{i=0}^{m} \theta_i \, P_{iT}(t) \;+\; x_t\,, \qquad t \;=\; 1,\, 2,\, \ldots\,, \tag{13}$$

with m indicating the order of the Chebyshev polynomial $P_{i,T}(t)$ defined as:

$$P_{0,T}(t) \;=\; 1,$$

$$P_{i,T}(t) \;=\; \sqrt{2}\cos\left(i\,\pi\,(t-0.5)/T\right), \qquad t \;=\; 1,\, 2,\, \ldots,\, T; \quad i \;=\; 1,\, 2,\, \ldots \tag{14}$$

(see Hamming (1973) and Smyth (1998) for a detailed description of these polynomials). Bierens (1997) and Tomasevic et al. (2009) argue that it is possible to approximate highly non-linear trends with polynomials of a rather low degree. This model includes the previous one noting that if $m = 0$ it contains an intercept, if $m = 1$ it includes a linear trend, and if $m > 1$ it becomes non-linear - the higher m is the less linear the approximated deterministic component becomes. Combining (13) with the second equation in (11) yields a linear model that can be estimated using least squares (see Cuestas and Gil-Alana, 2016).

[Insert Table 3 about here]

Table 3 displays the estimated coefficients of the Chebyshev polynomials in time along with the estimates of d for the case of uncorrelated errors (almost identical results were obtained with autocorrelated (Bloomfield) disturbances). To allow for some degree of nonlinearity, we set m equal to 3, therefore θ_2 and θ_3 are the coefficients corresponding to the nonlinear trends. Nonlinear behaviour is only found in the case of the quarterly real GDP series with the two nonlinear coefficients (θ_2 and θ_3) being statistically significant at the 5% level; θ_2 is also found to be significant in the case of the two real GDP per capita series (annual and quarterly) but not for annual real GDP. Further, the estimated values of d are all significantly higher than 1, ranging from 1.25 (quarterly real GDP) to 1.37 (annual real GDP per capita).[3]

Next we examine the possibility of a cyclical pattern in the data and for this purpose we consider a model specified as in (6): the coefficients of the first equation on the deterministic terms were not found to be statistically significant in any case, both with a linear time trend and when allowing for nonlinearities by using Chebyshev polynomials in time. Therefore, we estimate models for both the original and the demeaned series but without time trends, assuming in turn that u_t in (6) is a white noise process, an AR(1) process, and finally follows the exponential spectral specification of Bloomfield (1973). The results are presented in Tables 4, 5 and 6 respectively.

[Insert Tables 4, 5 and 6 about here]

In the case of white noise errors (Table 4), the estimated value of d for the original series is 10 years at the annual frequency, and 7 and 13 quarters respectively for real GDP and real GDP per capita at the quarterly frequency. The estimated value of r for the demeaned series is 12 at the annual frequency, and 7 and 8 respectively for real GDP and real GDP per capita at the quarterly frequency. It is also noteworthy that d_1 is

[3] Other nonlinear deterministic transformations produce insignificant coefficients in all cases.

systematically higher than d_2, which indicates that the long-run frequency is relatively more important than the cyclical one. Specifically, d_1 ranges between 0.55 (annual real GDP, original data) to 1.24 (quarterly real GDP per capita, demeaned data), while d_2 oscillates around 0, being significantly positive for the original series at the annual frequency as well as for both annual and quarterly real GDP per capita in the case of the demeaned series.

Table 5 displays the results under the assumption of AR(1) errors. In this case the estimated value of r is 10 for the four annual series, whilst it is 7 and 10 respectively for quarterly real GDP and real GDP per capita. Moreover, the estimated value of d_1 is much lower than in the previous case, and is not statistically different from zero for some of the original series. This might be a consequence of the competition with the AR(1) parameter in describing the degree of persistence in the long run structure of the data. For the demeaned series the values of d_1 are significant but smaller than those reported in Table 4. Besides, d_2 is now statistically significant in all cases, which implies the presence of a cyclical pattern.

Finally, Table 6 displays the results under the assumption that the error term follows the non-parametric specification of Bloomfield (1973). We consider these the most reliable evidence since this model allows for a certain degree of weak autocorrelation without affecting the estimation of the remaining parameters. The estimated value of r is now 10 for the two annual series, regardless of whether raw or demeaned data are used. The two differencing parameters are significantly different from zero in one of the four cases examined, with the value of the long-run parameter d_1 being around 0.60 and that of d_2 about 0.2 for the original data and slightly higher (about 0.3) for the demeaned series; for quarterly real GDP, r = 7, d_1 is around 0.7 and d_2 is close to zero for the original data but equal to 0.14 (and statistically significant) for the demeaned data; for quarterly real GDP per capita, r = 13, d_1 is equal to 0.66 for the original data and 0.91 for the demeaned data, and d_2 is statistically insignificant in both cases.

To summarise the main findings: first, there is evidence that when modelling GDP one should allow for long memory and fractional integration instead of restricting the differencing parameters to be either zero or one; second, both the zero (or long-run) frequency and the other non-zero (cyclical) frequencies play a role, at least in some cases. Awareness of the latter point is important for the purpose of evaluating the effects of shocks, which can affect not only the long run but also the cyclical structure. However, our modelling approach has the limitation of not being able to discriminate between the effects on the two components, since the error term is of a multiplicative nature and is based on both. Future work will aim to address this issue.

5. Conclusions

This paper proposes a new statistical model for macro series that captures both their long-run behaviour and their cyclical properties by including two poles in their spectrum, in addition to both linear and non-linear deterministic trends. The adopted framework is very general since it also allows for fractional degrees of integration and both long- and short-run memory components, and is suitable for modelling any macro series of interest. As an illustration, in this study it is applied to analyse four US real GDP series (annual and quarterly, per capita as well) and it is found to capture very well the stochastic properties of this series. In particular, both stochastic trends and stochastic cycles are found to be significant, both being mean-reverting, but the former being nonstationary and the latter stationary with cycles repeating themselves every 6 – 10 years. There is also evidence of persistence. Future work will extend this modelling framework with the aim of distinguishing between different types of shocks affecting trends and cycles separately while still allowing for a flexible degree of persistence.

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Figure 1: Original time series

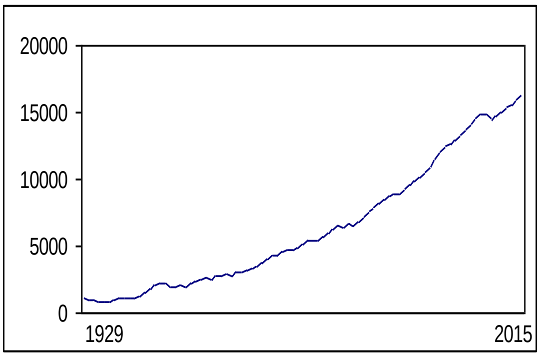
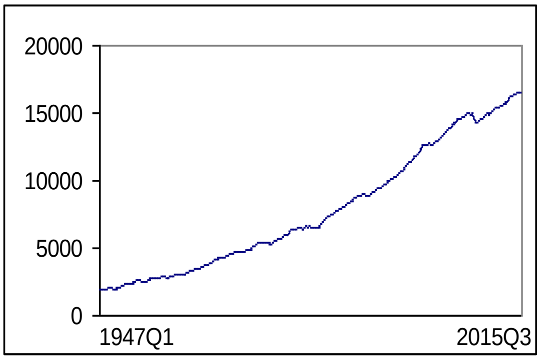
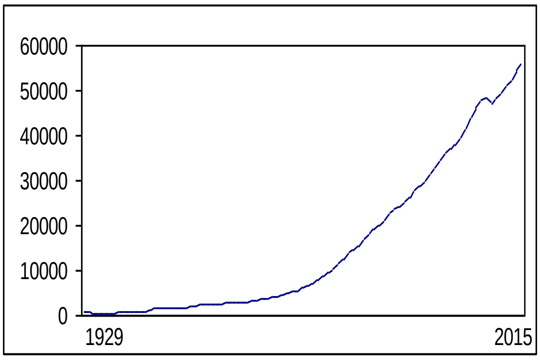
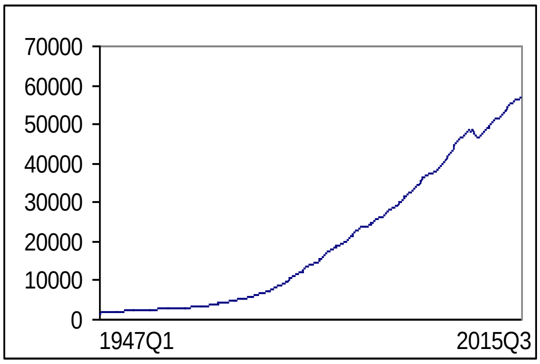
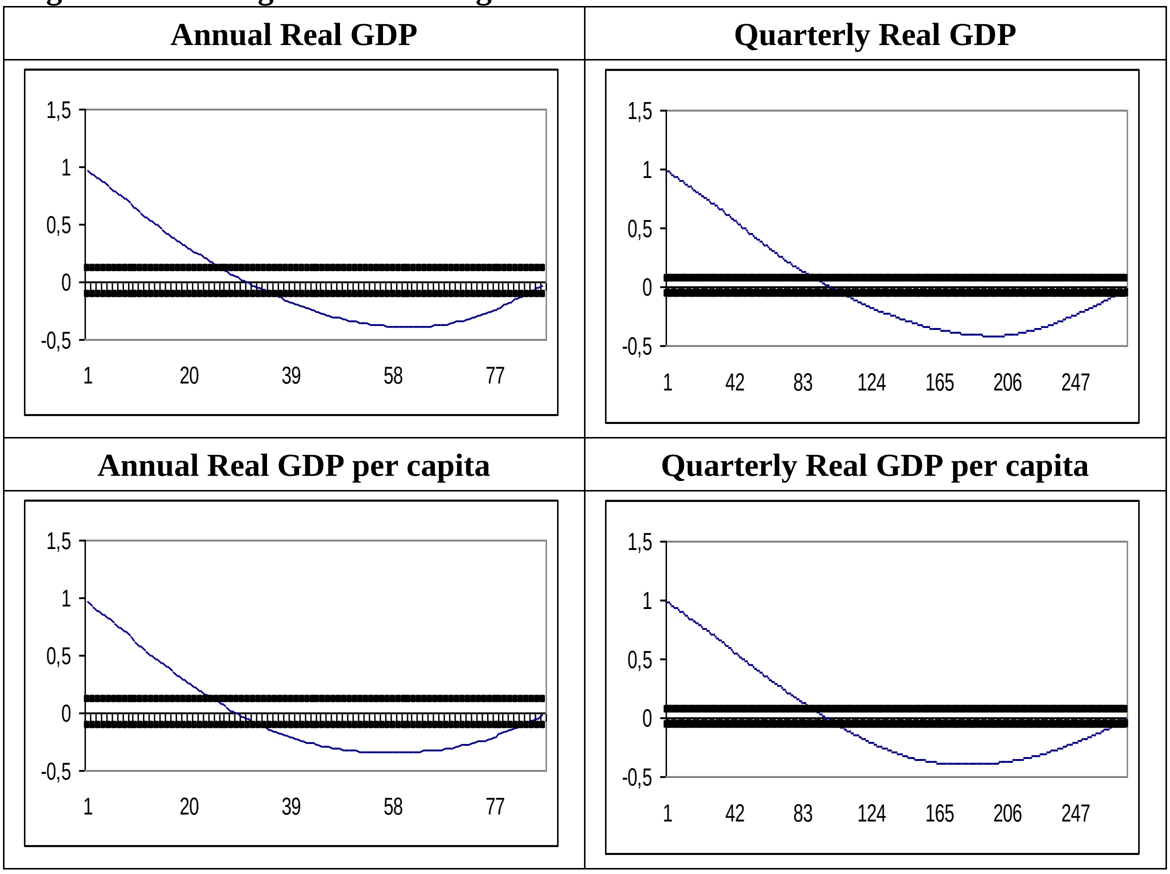
Annual Real GDP	Quarterly Real GDP
	
Annual Real GDP per capita	**Quarterly Real GDP per capita**
	

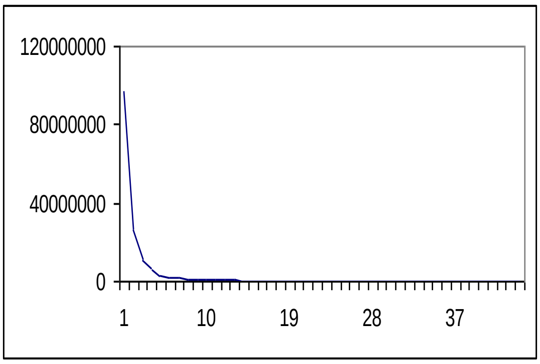
Figure 2: Correlograms of the original time series



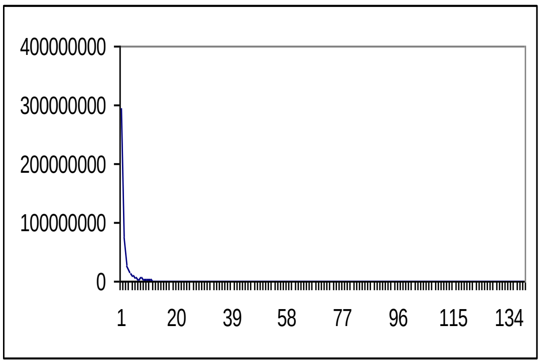
The thick lines refer to the 95% confidence bands for the null hypothesis of no autocorrelation.

Figure 3: Periodograms of the original time series

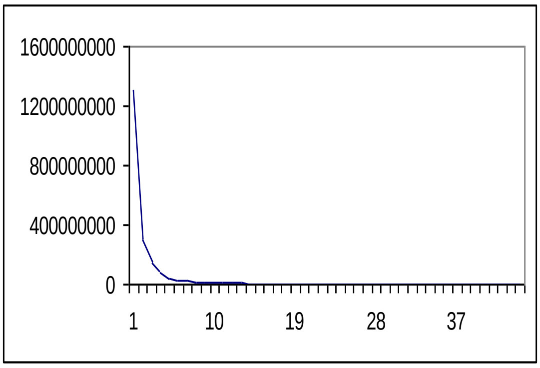
Annual Real GDP



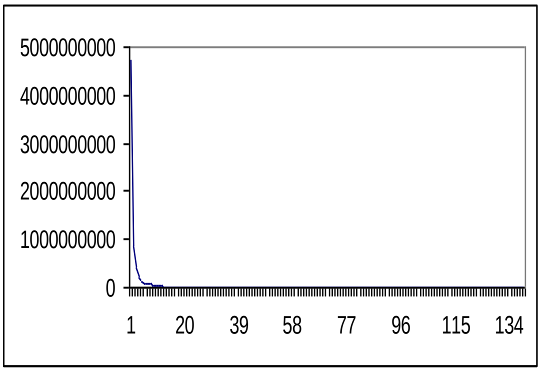
Quarterly Real GDP



Annual Real GDP per capita



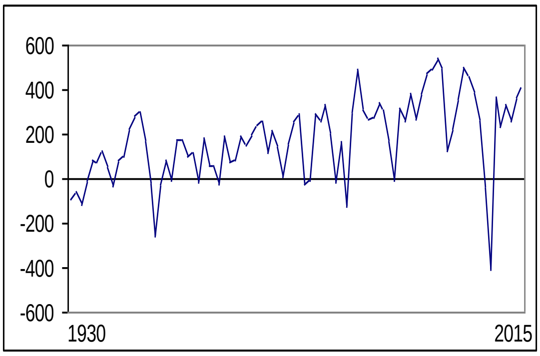
Quarterly Real GDP per capita



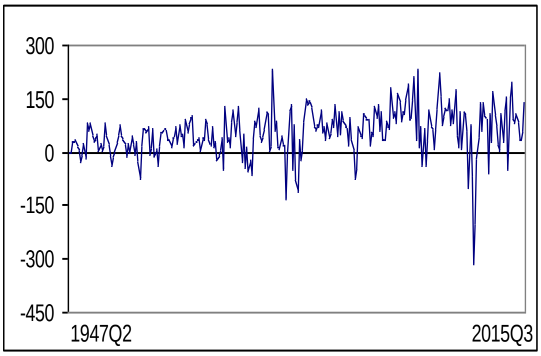
The horizontal axis refers to the discrete Fourier frequencies $\lambda j = 2\pi j/T$, $j = 1, 2, \ldots T/2$.

Figure 4: First differenced time series

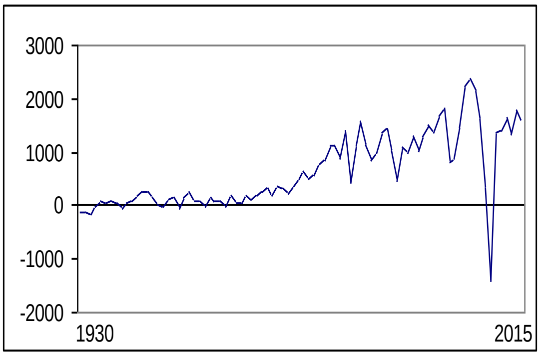
Annual Real GDP



Quarterly Real GDP



Annual Real GDP per capita



Quarterly Real GDP per capita

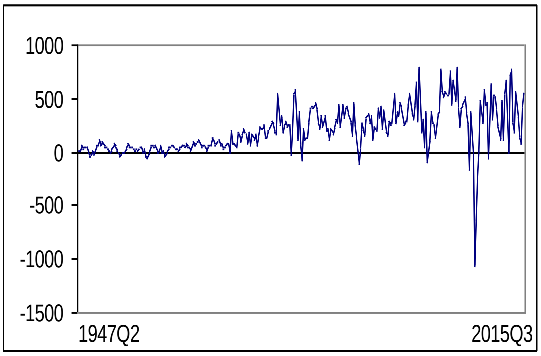
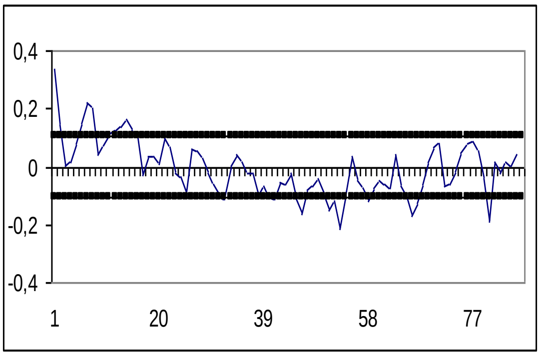
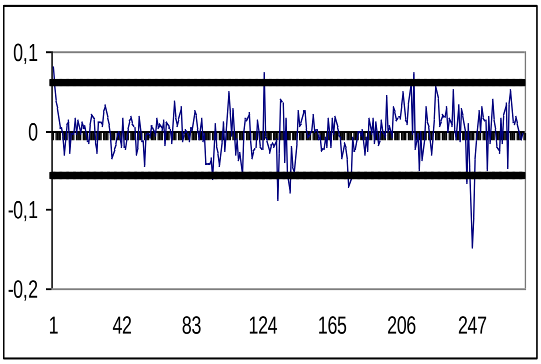
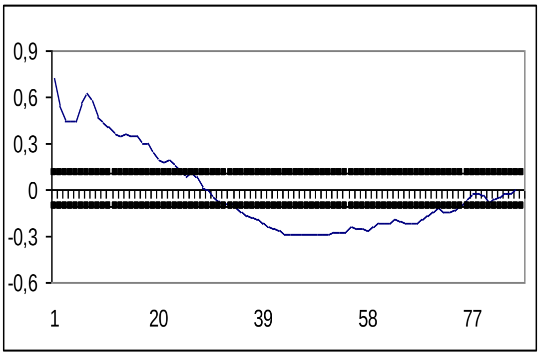
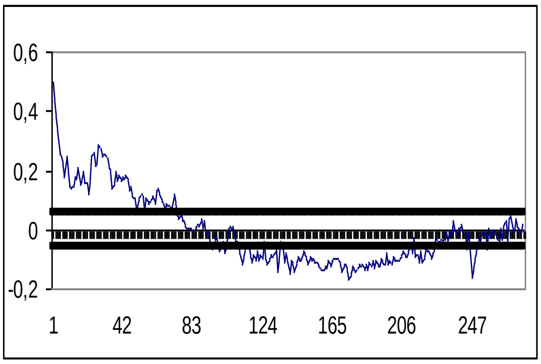
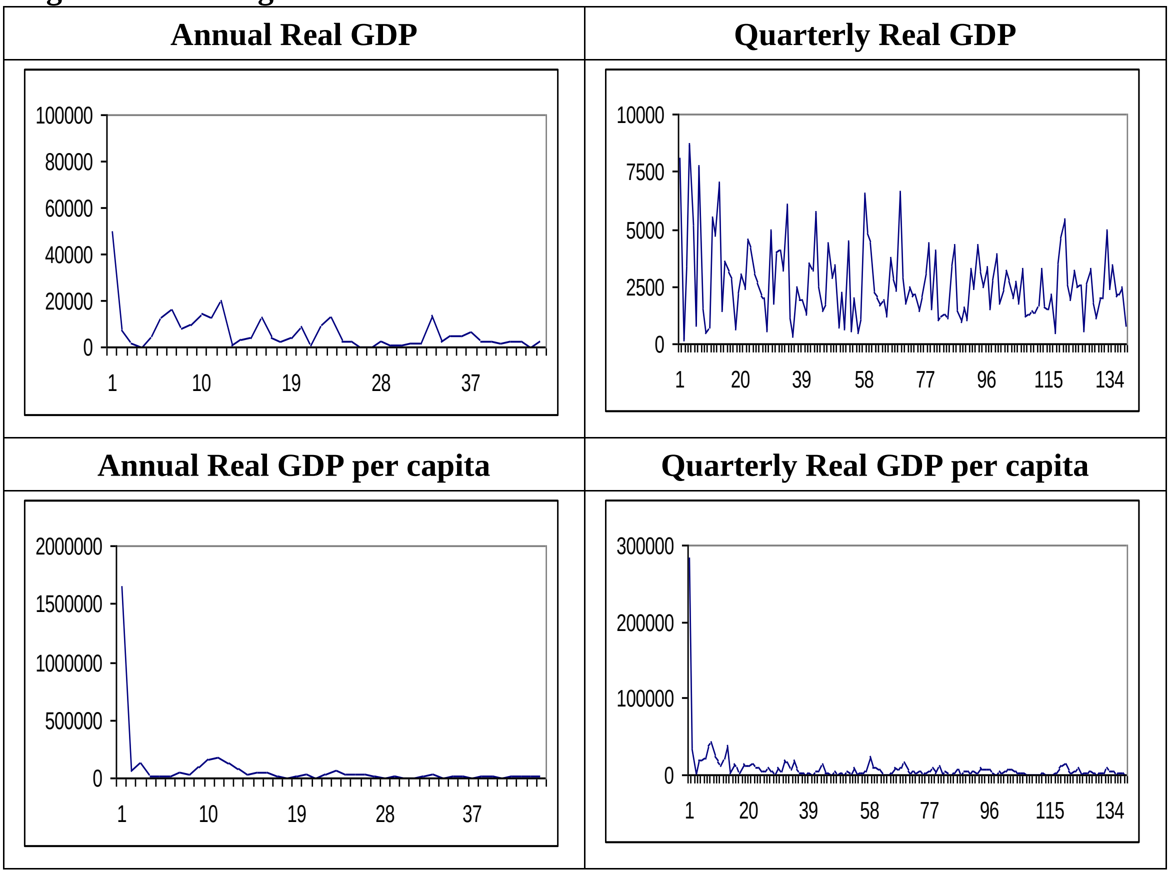


Figure 5: Correlograms of the first differenced time series

Annual Real GDP	Quarterly Real GDP
	
Annual Real GDP per capita	**Quarterly Real GDP per capita**
	

The thick lines refer to the 95% confidence bands for the null hypothesis of no autocorrelation.

Figure 6: Periodograms of the first differecend time series



The horizontal axis refers to the discrete Fourier frequencies λj = 2πj/T, j = 1, 2, …T/2.

Table 1: Estimated values of d with white noise errors

Series	No terms	An intercept	A linear trend
Annual real GDP	1.22 (1.11, 1.42)	1.31 (1.18, 1.58)	**1.36 (1.23, 1.58)**
Annual real GDP per cap	1.45 (1.34, 1.65)	**1.45 (1.35, 1.68)**	1.49 (1.39, 1.68)
Quarterly real GDP	1.09 (1.02, 1.18)	1.30 (1.22, 1.41)	**1.31 (1.24, 1.42)**
Quarterly real GDP per cap	1.33 (1.26, 1.42)	1.38 (1.31, 1.48)	**1.40 (1.34, 1.49)**

In bold, the selected models according to the deterministic terms.

Table 2: Estimated values of d with autocorrelated (Bloomfield) errors

Series	No terms	An intercept	A linear trend
Annual real GDP	1.08 (0.98, 1.25)	1.10 (1.00, 1.26)	**1.14 (1.00, 1.32)**
Annual real GDP per cap	1.29 (1.19, 1.43)	1.28 (1.18, 1.42)	**1.33 (1.22, 1.47)**
Quarterly real GDP	1.05 (0.97, 1.19)	1.22 (1.11, 1.43)	**1.26 (1.14, 1.44)**
Quarterly real GDP per cap	1.31 (1.22, 1.46)	1.35 (1.25, 1.14)	**1.39 (1.30, 1.55)**

In bold, the selected models according to the deterministic terms.

Table 3: Estimated values of d with white noise errors and nonlinear trends

Series	d	θ_0	θ_1	θ_2	θ_3
Annual real GDP	1.31 (1.12, 1.52)	6727.29 (2.92)	-3727.39 (-2.51)	665.09 (1.25)	-328.19 (-1.05)
Annual real GDP per cap	1.37 (1.23, 1.49)	13192.94 (1.47)	-11626.58 (-1.99)	3762.34 (1.91)	-271.74 (-0.24)
Quarterly real GDP	1.25 (1.01, 1.57)	6202.24 (3.19)	-4108.28 (-3.33)	1013.71 (2.19)	-524.42 (-1.89)
Quarterly real GDP per cap	1.39 (1.20, 1.68)	10072.98 (2.20)	-10468.05 (-1.93)	5500.76 (3.00)	-1523.63 (-1.44)

Table 4: Estimated coefficients with white noise errors

Series	Original data			De-meaned data		
	r	d_1	d_2	r	d_1	d_2
Annual real GDP	10	0.55^*	0.20^*	12	0.75^*	0.01
Annual real GDP per cap	10	0.64^*	0.15^*	12	0.63^*	0.36^*
Quarterly real GDP	7	0.69^*	-0.01	7	0.73^*	0.14^*
Quarterly real GDP per cap	13	0.66^*	0.04	8	1.24^*	-0.13

*: Statistical significance at the 5% level.

Table 5: Estimated coefficients with AR(1) errors

Series	Original data			De-meaned data		
		d_1	d_2	r	d_1	d_2
Annual real GDP	10	0.01	0.32^*	10	0.26^*	0.12^*
Annual real GDP per cap	10	0.01	0.37^*	10	0.29^*	0.14^*
Quarterly real GDP	7	0.65^*	0.40^*	7	0.25^*	0.11^*
Quarterly real GDP per cap	13	0.00	0.29^*	13	0.58^*	0.46^*

*: Statistical significance at the 5% level.

Table 6: Estimated coefficients with Bloomfield-type errors

Series	Original data			De-meaned data		
	r	d_1	d_2	r	d_1	d_2
Annual real GDP	10	0.55^*	0.20^*	10	0.60^*	0.32^*
Annual real GDP per cap	10	0.63^*	0.17^*	10	0.63^*	0.26^*
Quarterly real GDP	7	0.69^*	-0.01	7	0.73^*	0.14^*
Quarterly real GDP per cap	13	0.66^*	0.04	13	0.91^*	-0.03

*: Statistical significance at the 5% level.