Invariant β-ensembles and the Gauss-Wigner crossover

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Gaussian Orthogonal/Unitary Ensembles

A random symmetric (resp. hermitian) matrix H of size $N \times N$ belongs to the GOE (resp. GUE) if

$$P(dH) = \frac{1}{Z_N} \exp\left(-\frac{1}{2}\mathrm{Tr}(H^\dagger H)\right) dH \,.$$

For any $O \in \mathcal{O}(N)$ (resp. $\in \mathcal{U}(N)$),

$$OHO^\dagger \overset{(law)}{=} H \,.$$

Hence H have Haar distributed eigenvectors.
So the eigenvalues are independent of the eigenvectors and distributed according to the joint distribution with density (Dyson, 1962)

$$P_\beta(\lambda_1, \cdots, \lambda_N) = \frac{1}{Z_N^\beta} \prod_{i<j} |\lambda_i - \lambda_j|^\beta \exp\left(-\frac{1}{2}\sum_{i=1}^N \lambda_i^2\right) \quad (1)$$

with $\beta = 1$ in the real symmetric case (resp. $\beta = 2$ in the complex hermitian case).

Dyson Brownian motion

Consider a sym. (resp. herm.) $M(t)$ s. t. for $i \le j$,

$$dM_{ij}(t) = -\frac{1}{2} M_{ij}(t)dt + dB_{ij}(t)$$

where $B_{ii}(t)$ is a real Brownian motion and for $i < j$, $B_{ij}(t) = \frac{1}{\sqrt{2}}\left(B^1_{ij}(t) + \sqrt{-1}\,(\beta-1)B^2_{ij}(t)\right)$, with B^1_{ij}, B^2_{ij} indep. BMs. Nice thing is $t \to +\infty$:

$$M(t) \overset{(law)}{\Rightarrow} H \in \text{GOE} \quad (\text{resp. GUE})\,.$$

Using perturbation theory,

$$d\lambda_i(t) = -\frac{1}{2}\lambda_i(t)dt + db_i(t) + \frac{\beta}{2}\sum_{j\neq i}\frac{dt}{\lambda_i(t)-\lambda_j(t)}$$

where the b_i are independent BMs. For $\beta \ge 1$ (here in fact $\beta = 1$ or 2), there are almost surely no collisions between the particles.

Fokker Planck Eq. for the transition density $P(\lambda_1, \cdots, \lambda_N; t)$:

$$\frac{\partial P}{\partial t} = -\sum_{i=1}^N \frac{\partial}{\partial \lambda_i}\left[\left(-\frac{\lambda_i}{2} + \frac{\beta}{2}\sum_{j\neq i}\frac{1}{\lambda_i - \lambda_j}\right)P\right] + \frac{1}{2}\sum_{i=1}^N \frac{\partial^2 P}{\partial \lambda_i^2}\,.$$

The stationary solution ($\frac{\partial P}{\partial t} = 0$) is indeed P_β of Eq. (1).

Interpolation process

For $\beta = 1$ and $\beta = 2$, GOE and GUE are rotationally invariant matrix models with eigenvalues distributed according to the joint pdf P_β.

Purpose: construct a rotationally invariant matrix model with eigenvalues distributed as P_β for all $0 < \beta < 2$.

Dumitriu & Edelman tridiagonal ensembles

Nice explicit matrix model

$$H^\beta = \frac{1}{\sqrt{2}}\begin{pmatrix} g_1 & \chi_{(N-1)\beta} & & & \\ \chi_{(N-1)\beta} & g_2 & \chi_{(N-2)\beta} & & \\ & \ddots & \ddots & \ddots & \\ & & \chi_{2\beta} & g_{N-1} & \chi_\beta \\ & & & \chi_\beta & g_N \end{pmatrix}$$

where the g_k are gaussian variables and the χ_k are indep. chi random variables indexed by their shape parameter.

Eigenvalues distributed according to P_β for any $\beta > 0$ but obviously not rotationally invariant !

Dynamical procedure for an invariant model

Idea: Alternate free addition (of GO/UE matrix) and addition of independent random variables (by adding a diagonal matrix with i.i.d entries).

Divide time into small intervals $[k/n; (k+1)/n]$, and for each k toss a coin ε^n_k such that $P[\varepsilon^n_k = 1] = p = 1 - P[\varepsilon^n_k = 0]$.

$$dM_n(t) = -\frac{1}{2}M_n(t)dt + \epsilon^n_{[nt]}dH_t + (1-\epsilon^n_{[nt]})dY_t$$

dH_t: Dyson Brownian motion increment.

dY_t: sym. matrix co-diagonalizable with $\mathbf{M}_n(t)$ with a spectrum given by N independent real Brownian increments.

$M_n(t)$ is rotationally invariant !

Our result

The eigenvalues process $(\lambda^n_1(t), \cdots, \lambda^n_N(t))$ of $M_n(t)$ converges in law when $n \to \infty$ to the solution $(\lambda_1(t), \cdots, \lambda_N(t))$ of the following SDE

$$d\lambda_i = -\frac{1}{2}\lambda_i dt + \frac{p}{2}\sum_{j\neq i}\frac{dt}{\lambda_i - \lambda_j} + db_i\,.$$

The stationary joint pdf of the λ_i is P_β with $\beta = p$.

For $p < 1$, difficulties arise as the particles collide.

A collision leads to a complete randomization of the eigenvectors within their two dimensional subspace.

Density of eigenvalues

- If $p > 0$ does not depend on N, then the large N *stationary* density is

$$\rho_N(\lambda) = \frac{1}{\pi p N}\sqrt{2pN - \lambda^2}\,\mathbf{1}_{\{|\lambda| \le \sqrt{2pN}\}}.$$

  Wigner semicircle density.
- If $p = 0$, the particles evolve as independent BM, thus

$$\rho(\lambda) = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{\lambda^2}{2}}\,.$$

Let $G(z,t) = \frac{1}{N}\sum_{i=1}^N E[\frac{1}{\lambda_i(t) - z}]$.
Using Itô's formula,

$$2\frac{\partial G}{\partial t} = \frac{pN}{2}\frac{\partial G^2}{\partial z} + \frac{\partial zG}{\partial z} + \frac{2-p}{2}\frac{\partial^2 G}{\partial z^2}\,.$$

Further results: regime $p = 2c/N$

For $p = 2c/N$, the stationary solution satisfies

$$c\, G^2_\infty(z) + G'_\infty(z) + zG_\infty(z) + 1 = 0\,.$$

With Riccati transform $G(z) = u'(z)/cu(z)$, this can be solved ! This leads to

$$\rho_c(\lambda) = \frac{1}{\sqrt{2\pi}\Gamma(1+c)}\frac{1}{|D_{-c}(i\lambda)|^2}$$

where

$$D_{-c}(z) = \frac{e^{-z^2/4}}{\Gamma(c)}\int_0^\infty \mathrm{d}x\, e^{-zx-\frac{x^2}{2}}x^{c-1}\,.$$

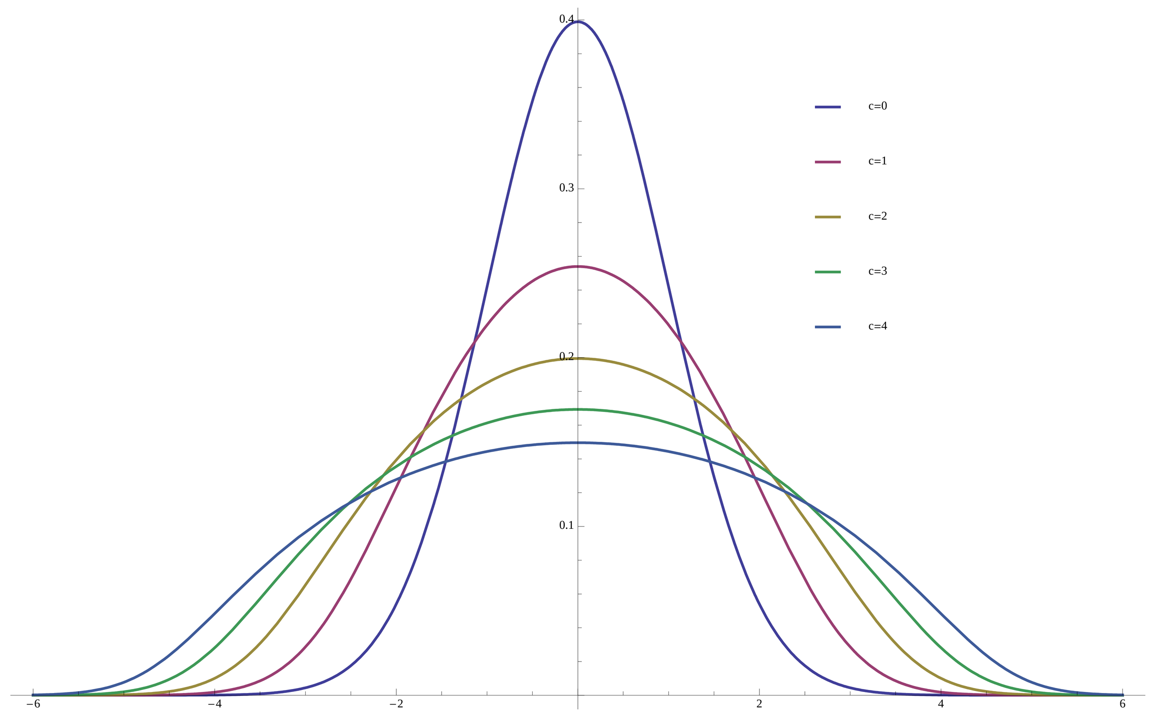

Figure 1: Graph of ρ_c for $c = 0, 1, 2, 3, 4$.

Extension to Wishart matrices with Satya N. Majumdar and Pierpaolo Vivo in [3]

Interpolation matrix processes between CIR processes and Wishart processes, interpolation for the density between Marčenko Pastur distribution and Gamma law through Witthaker functions. Corrections to the Marčenko pastur law for large but finite N.

References

[1] R. Allez and A. Guionnet, *A diffusive matrix model for invariant β-ensembles*, submitted, (2012).

[2] R. Allez, J.-P. Bouchaud and A. Guionnet, *Invariant β-ensembles and the Gauss-Wigner crossover*, Phys. Rev. Lett. **109**, 094102 (2012).

[3] R. Allez, J.-P. Bouchaud, S. N. Majumdar, P. Vivo, *Invariant β-Wishart ensembles, crossover densities and asymptotic corrections to the Marčenko-Pastur law*, J. Phys. A: Math. Theor. **46** 015001 (2013).