Painlevé III and a Matrix Model with Singular Weight

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An average over the GUE - Introduction and Motivation

We want to find the following average over the Gaussian Unitary Ensemble (GUE):

$$E_N(z,t) := \frac{1}{N!}\int_{\mathbb{R}^N} \prod_{j=1}^{N} \exp\left(-\frac{z^2}{2x_j^2} + \frac{t}{x_j} - \frac{x_j^2}{2}\right) \prod_{1\le j<k\le N} |x_k - x_j|^2 d^N x \tag{1}$$

Integrals such as this are known as **Partition Functions** and are extensively studied as they contain information on correlations of eigenvalues. This particular integral is of interest because certain statistics of eigenvalues of the GUE which are related to the study of zeros of the Riemann zeta function are encoded in the power series in t of $E_N(z,t)$.

Mezzadri, Mo (2009): For $c_1 N^{-\frac{1}{2}} < z < c_2 N^{\frac{1}{4}}$,

$$E_N(z,t) = B_N \exp\left(\frac{z^2}{4} - \frac{9}{2^{\frac{10}{3}}}\left(N^{\frac{2}{3}} z^{\frac{4}{3}} - 1\right) + \frac{t^2 N^{\frac{1}{3}}}{2^{\frac{5}{3}} z^{\frac{4}{3}}}\right)(1+o(1)) \quad N\to\infty \qquad B_N := \int_{\mathbb{R}^N} \prod_{j=1}^{N} e^{-\frac{1}{2Nx_j^2}} P_{GUE}(x_1,\dots,x_N) d^N x \tag{2}$$

In the double scaling limit $zN^{\frac{1}{2}} \to 0$ and $N\to\infty$ this result is no longer valid, a phase transition occurs and $E_N(z,t)$ is found in terms of a solution to the Painlevé III equation.

This is the first time that PIII has appeared in the study of double scaling limits in Random Matrix Models. It seems as though this is due to the essential singularity in the weighting function

Connection to a Riemann-Hilbert problem for Orthogonal Polynomials on the Real Line

Setting $(u_1, u_2) = (\sqrt{N}t, Nz^2)$ gives a rescaled partition function with weight

$$w_N(y) = \exp\left(-N\left(\frac{u_2}{2N^3y^2} + \frac{y^2}{2}\right) + \frac{u_1}{Ny}\right) \qquad y = x\sqrt{N} \tag{3}$$

Then for monic polynomials $\pi_k(y)$ orthogonal with respect to $w_N(y)$,

$$\int_{\mathbb{R}} \pi_j(y)\pi_k(y) w_N(y)\, dy = h_k \delta_{jk}$$

and matrix

$$Y(y) := \begin{pmatrix} \pi_N(y) & \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\pi_N(q)w_N(q)}{q-y}dq \\ -\frac{2\pi i}{h_{N-1}}\pi_{N-1}(y) & -\frac{1}{h_{N-1}}\int_{-\infty}^{\infty} \frac{\pi_{N-1}(q)w_N(q)}{q-y}dq \end{pmatrix} \tag{4}$$

Lemma (Bertola, Eynard and Harnad):

$$\frac{\partial \log E_N}{\partial u_1} = -\frac{1}{4\pi i N}\oint_{y=0} \mathrm{Tr}\left(Y^{-1}(y)Y'(y)\sigma_3\right)\frac{dy}{y} \tag{5a}$$

$$\frac{\partial \log E_N}{\partial u_2} = \frac{1}{8\pi i N^2}\oint_{y=0} \mathrm{Tr}\left(Y^{-1}(y)Y'(y)\sigma_3\right)\frac{dy}{y^2} \tag{5b}$$

where $\sigma_3 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$.

This Lemma allows us to find an asymptotic expansion for $E_N(z,t)$ using the asymptotics of $Y(y)$. For z in the required range, this is done using Deift-Zhou steepest descent analysis.

PROBLEM: In the double scaling limit, the asymptotic analysis analysis used to obtain (2) breaks down - we need different techniques to study the asymptotics of $Y(y)$ near the origin.

Main Result

Theorem: Set $u_1 = 0$, $r = \sqrt{u_2}$ and let $v_Y(r)$ be a solution of the PIII equation

$$v_Y'' = \frac{(v_Y')^2}{v_Y} - \frac{v_Y'}{r} + (-1)^N \frac{v_Y^2}{r} + v_Y^3$$

Then the canonical variables

$$P = \frac{ir}{2v_Y(r)} + \frac{iu_2 v_Y^2(r)}{4} - (-1)^N \frac{iu_2 v_Y'(r)}{4}$$

$$Q = \frac{(-1)^N i}{r v_Y(r)} - (-1)^N \frac{i v_Y^2(r)}{2} + \frac{i v_Y'(r)}{2}$$

solve the Hamilton equations

$$\partial_{u_2} P = -\frac{\partial(H+h)}{\partial Q} \qquad \partial_{u_2} Q = \frac{\partial(H+h)}{\partial P}$$

where

$$\frac{\partial}{\partial z} E_N(z,t) = -2zNH + O(1)$$

Painlevé Type Differential Equations

Away from the origin: $Y(y) \mapsto S^{\infty}(y)$ - has nicer asymptotics than $Y(y)$.

Near the origin: $Y(y) \mapsto \Psi(\zeta)$ - **constant jumps** and singularities at $\zeta = 0, \infty$:

$$\begin{aligned} \Psi(\zeta) &= \hat{\Psi}_0(\zeta) e^{\left(-\frac{u_2}{4\zeta^2} + \frac{u_1}{2\zeta}\right)\sigma_3} & \zeta &\to 0 \\ \Psi(\zeta) &= \hat{\Psi}_\infty(\zeta) e^{i\zeta\sigma_3} & \zeta &\to \infty \end{aligned} \tag{6}$$

where $\zeta = N(y + O(y^3))$.

$$Y(y) = e^{\frac{Nl\sigma_3}{2}} S^{\infty}(y) e^{-\frac{iN\pi}{2}\sigma_3} \hat{\Psi}_0(\zeta) e^{\frac{Ny^2}{4}\sigma_3} \qquad \zeta \to 0 \tag{7}$$

Key Observation: Define $A(\zeta)$, $B_{1,2}(\zeta)$ to be

$$A(\zeta) := \frac{\partial \Psi(\zeta)}{\partial \zeta}\Psi^{-1}(\zeta) \qquad B_{1,2}(\zeta) := \frac{\partial \Psi(\zeta)}{\partial u_{1,2}}\Psi^{-1}(\zeta) \tag{8}$$

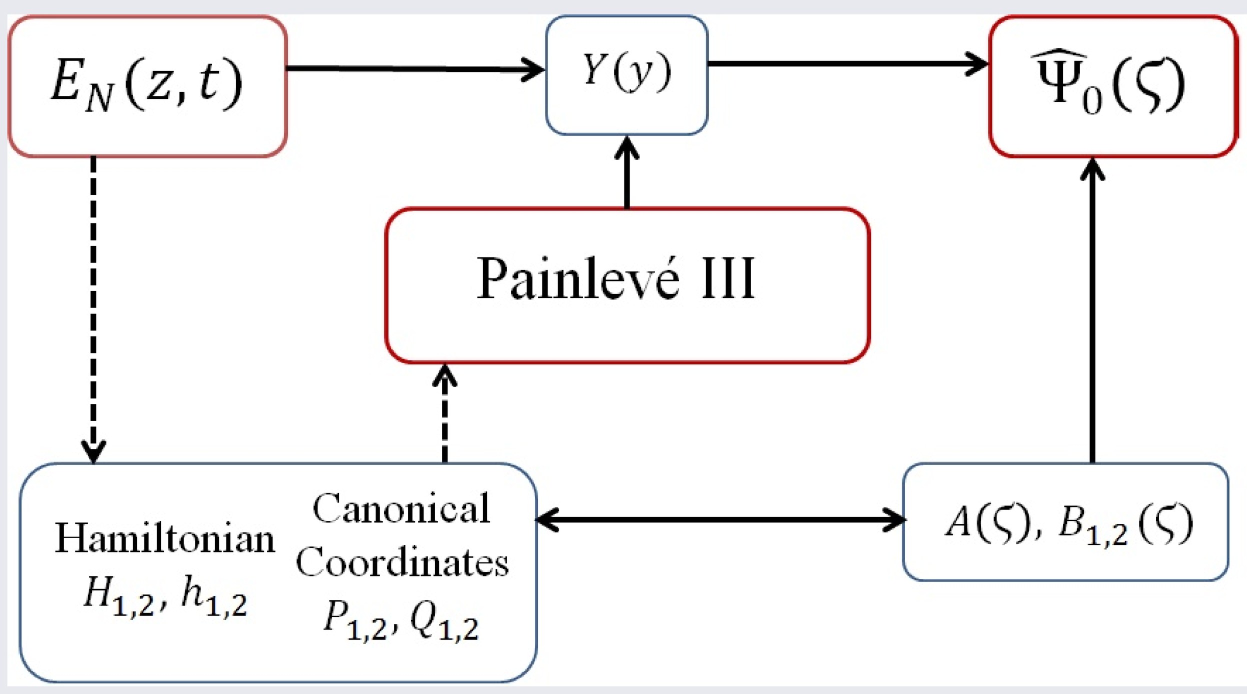
By the jump discontinuities and singularity structure of $\Psi(\zeta)$, the matrices $A(\zeta)$, $B_{1,2}(\zeta)$ are rational functions of ζ, with poles at $\zeta = 0, \infty$.

A map of the connection:

- $A(\zeta)$, $B_{1,2}(\zeta)$ give rise to a system of PDEs in u_1 and u_2.
- Theory of isomonodromic deformations tells us this is equivalent to a Hamiltonian system of ODEs
- Equations (6) and (8) provide a link between $A(\zeta)$, $B_{1,2}(\zeta)$ and $\hat{\Psi}_0(\zeta)$.
- Hence, with (5) and (7) we find $E_N(z,t)$ in terms of the Hamiltonian system related to $A(\zeta)$, $B_{1,2}(\zeta)$.

...now to find the connection to PIII!

Right: A map of the connection between $E_N(z,t)$ and PIII. The solid lines show the explicit connections, and the dotted lines show the derived connections via $\hat{\Psi}_0(\zeta)$.



Connection to Painlevé III

Need a result by **Chen and Its (2010):** For $\alpha > -1$, $s > 0$ consider the weight

$$W_\alpha(x) = \exp(-s/x - x)\, x^\alpha, \qquad x \in \mathbb{R}_+, \tag{9}$$

polynomials $P_n(x)$ such that

$$\int_{\mathbb{R}_+} P_n(x)P_m(x)W_\alpha(x)dx = h_n^{(p)}\delta_{mn}$$

and matrix function

$$X(x) := \begin{pmatrix} P_n(x) & \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{P_n(q)W_\alpha(q)}{q-x}dq \\ -\frac{2\pi i}{h_{n-1}^{(p)}}P_{n-1}(x) & -\frac{1}{h_{n-1}^{(p)}}\int_{-\infty}^{\infty} \frac{P_{n-1}(q)W_\alpha(q)}{q-x}dq \end{pmatrix} \tag{10}$$

Define

$$v_X(\omega) := \frac{h_n^{(p)}}{2\pi i\omega X_{11}(0)X_{12}(0)} \qquad \omega = \sqrt{s} \tag{11}$$

Then $v_X(\omega)$ satisfies the PIII equation

$$v_X'' = \frac{(v_X')^2}{v_X} - \frac{v'}{\omega} - \frac{1}{\omega}\left(4\alpha v_X^2 + 4(2n+1+\alpha)\right) + 4v_X^3 - \frac{4}{v_X} \tag{12}$$

The change of variables

$$y^2 = \frac{2x}{N}, \quad s = \frac{u_2}{4N}, \quad n = \left\lfloor \frac{N}{2} \right\rfloor$$

gives

$$v_X(\omega) = \begin{cases} \frac{\sqrt{N}e^{Nl}}{i\sqrt{u_2}Y_{11}\partial_\zeta Y_{12}}\left(1 + O(N^{-\frac{1}{2}})\right) & N \text{ even} \\ \frac{\sqrt{N}e^{Nl}}{i\sqrt{u_2}\partial_\zeta Y_{11} Y_{12}}\left(1 + O(N^{-\frac{1}{2}})\right) & N \text{ odd} \end{cases} \tag{13}$$

By the asymptotic behaviour of $Y(y)$, $v_X(\omega) = O(\sqrt{N})$ as $N \to \infty$. So,

$$v_X(\omega) := \sqrt{N} v_Y(\sqrt{u_2})\left(1 + O\left(N^{-\frac{1}{2}}\right)\right)$$

and $v_Y(\sqrt{u_2})$ will satisfy a PIII equation associated to $Y(y)$.

We use equation (7) to write $Y_{11}(\zeta)$, $Y_{12}(\zeta)$ and their derivatives in terms of $\Psi(\zeta)$.

This gives us all the required information to make the connection between PIII and the double scaling limit of $E_N(z,t)$.