Universality in Fixed Trace Ginibre Ensembles

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Introduction

- The **Ginibre ensemble** defined by

$$\mathbf{Z}_{\mathcal{G}} := \int_{\mathbb{C}} e^{-\mathrm{Tr}[\mathbf{GG}^{\dagger}]}\, \mu(\mathrm{d}\mathbf{G})$$

deals with quadratic matrices of size $\mathbf{N} \in \mathbb{N}$ with no hermitian or unitary conditions imposed and was introduced by Ginibre [1] in 1965. Here we are interested in the case of complex matrices $\mathbf{G} \in \mathbb{C}^{\mathbf{N}\times\mathbf{N}}$ where $\mathbf{G}_{ij} \in \mathcal{N}(\mathbf{0},\mathbf{1})$ for the real and imaginary part, respectively.
Using the Schur decomposition, computing the Jacobian, following the method of orthogonal polynomials (monoms) and applying the famous Dyson theorem leads to the $\mathbf{k}$-point function in terms of the determinant of the kernel. In particular the angle-independent normalized eigenvalue distribution in the complex plane is given by the one point function

$$\mathbf{R}_{1,\mathcal{G}}(\mathbf{r}) = \frac{\mathbf{1}}{2\pi\mathbf{N}} e^{-\mathbf{r}^2} \sum_{\mathbf{k}=0}^{\mathbf{N}-1} \frac{\mathbf{r}^{2\mathbf{k}}}{\mathbf{k}!}.$$

The initial interests in non-hermitian matrices goes back to the theory of scattering quantum chaotic systems [2] whose particles can escape at a given energy to infinity or come from infinity. The random matrix description comes with the so called Heidelberg approach where the description of the theory is based on an effective Hamiltonian which is non-hermitian.

- The **fixed trace ensemble** is defined by

$$\mathbf{Z}_{\delta} := \int_{\mathbb{C}} \delta\left(\mathbf{t} - \mathrm{Tr}\left[\mathbf{GG}^{\dagger}\right]\right)\ \mu(\mathrm{d}\mathbf{G}),$$

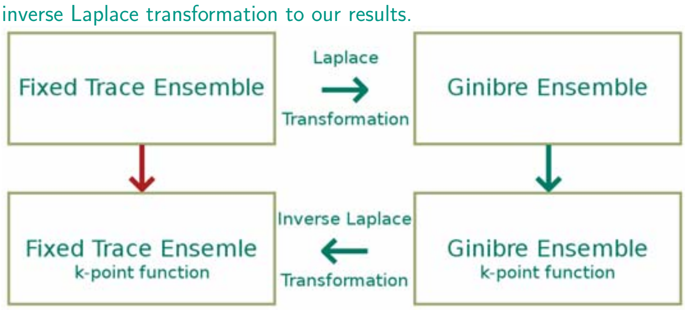
where $\mathbf{G}$ preserve the above properties.
The fixed trace ensemble, in particular for covariance matrices $\mathbf{W} = \mathbf{XX}^{\dagger}$, $\mathbf{X} \in \mathbb{C}^{\mathbf{N}\times\mathbf{M}}$ can be used to describe the entanglement of a bipartite quantum system [3]. Let us consider a system A in which we are interested in and B as the environment. The bipartite Hilbert space is give by $\mathcal{H}^{\mathbf{N}\times\mathbf{M}} = \mathcal{H}_{\mathbf{A}}^{\mathbf{N}} \otimes \mathcal{H}_{\mathbf{B}}^{\mathbf{M}}$ and w.l.o.g. we can assume that $\mathbf{N} \leq \mathbf{M}$. Let us furthermore consider a random pure state restricted by $\mathrm{Tr}\,[\rho] = \mathbf{1}$, then it turns out that the eigenvalues of system A are the so called Schmidt eigenvalues and can be obtained by the Hilbert-Schmidt distribution $\delta\left(\mathbf{1} - \mathrm{Tr}\left[\mathbf{XX}^{\dagger}\right]\right)$.

Map between Ginibre and Fixed Trace

In comparison to the Ginibre ensemble where the eigenvalues in the weight function can be separated from each other, in the fixed trace case this is no longer true. Here we are not able to apply the method of orthogonal polynomials due to the weight function which is given by the Dirac delta distribution.
However we can find a map between the fixed trace and the Ginibre ensemble via the Laplace transformation

$$\mathcal{L}\left\{\delta\left(\mathbf{t} - \mathrm{Tr}\left[\mathbf{GG}^{\dagger}\right]\right)\right\}(\mathbf{s}) = \int_{\mathbf{0}}^{\infty} \mathrm{d}\mathbf{t}\ \delta\left(\mathbf{t} - \mathrm{Tr}\left[\mathbf{GG}^{\dagger}\right]\right) e^{-\mathbf{st}} = e^{-\mathbf{s}\mathrm{Tr}[\mathbf{GG}^{\dagger}]}.$$

The only thing we need to do is to extend the Ginibre ensemble by the arbitrary integration parameter $\mathbf{s}$. Given this map, we can stay in the Laplace space, which correspond to the Ginibre ensemble, calculate the $\mathbf{k}$-point function and went back to the fixed trace ensemble by applying the inverse Laplace transformation to our results.



Finite N behavior

Following the above approach we can obtain the normalized angle-independent eigenvalue distribution by computing the inverse Laplace transformation of the one point function given by the Ginibre ensemble. We obtain by setting $\mathbf{t} = \mathbf{1}$

$$\mathbf{R}_{1,\delta}(\mathbf{r}) = \frac{(\mathbf{N}^2-\mathbf{1})\,\Theta\left(\mathbf{1}-\mathbf{r}^2\right)}{2\pi\mathbf{N}} \sum_{\mathbf{k}=0}^{\mathbf{N}-1} \binom{\mathbf{N}^2-2}{\mathbf{k}} \mathbf{r}^{2\mathbf{k}} \left(\mathbf{1}-\mathbf{r}^2\right)^{\mathbf{N}^2-2-\mathbf{k}}.$$

Universal large N-limit

In the eigenvalue distribution $\mathbf{R}_{1,\delta}$ we can consider the expression under the sum as the binomial distribution, due to the fact that the radius $\mathbf{r}^2$ is restricted by theta function to the interval $[\mathbf{0},\mathbf{1}]$, hence for the large $\mathbf{N}$-limit we can follow the common Poisson approach. By defining $\lambda := \mathbf{rN}$, we obtain

$$\lim_{\mathbf{N}\to\infty} \binom{\mathbf{N}^2-2}{\mathbf{k}} \left(\frac{\lambda^2}{\mathbf{N}^2}\right)^{\mathbf{k}} \left(\mathbf{1}-\frac{\lambda^2}{\mathbf{N}^2}\right)^{\mathbf{N}^2-2-\mathbf{k}} = e^{-\lambda^2}\frac{\lambda^{2\mathbf{k}}}{\mathbf{k}!}.$$

Summarized the limiting eigenvalue distribution for large matrices can be expressed in the following way

$$\lim_{\mathbf{N}\to\infty} \mathbf{R}_{1,\delta}(\mathbf{r}) = (\mathbf{N}^2-\mathbf{1})\,\Theta\left(\mathbf{1}-\mathbf{r}^2\right) \underbrace{\frac{\mathbf{1}}{2\pi\mathbf{N}} e^{-(\mathbf{rN})^2} \sum_{\mathbf{k}=0}^{\mathbf{N}-1} \frac{(\mathbf{rN})^{2\mathbf{k}}}{\mathbf{k}!}}_{=\mathbf{R}_{1,\mathcal{G}}(\mathbf{rN})},$$

such that we recover the scaled one point function of the Ginibre ensemble. Thus we can conclude that the eigenvalue distribution of the fixed trace ensemble is universal in the bulk.

Extensions and Generalizations

- **k-point function**: We computed the $\mathbf{k}$-point function for finite $\mathbf{N}$. It turns out that it is no determinantal point process. Moreover the universal behavior in the bulk, as shown for the one point function, can be extended to the general case of the $\mathbf{k}$-point function.
- **Normal matrices**: Our computations holds also in the case of considering normal matrices $\mathbf{N} \in \mathbb{C}^{\mathbf{N}\times\mathbf{N}}$, $\mathbf{NN}^{\dagger} = \mathbf{N}^{\dagger}\mathbf{N}$. The difference in the Ginibre case to non-hermitian matrices is just in terms of a constant, which for normal matrices disappear due to the diagonalizability. However, it turns out that the functional relationship for finite $\mathbf{N}$ in the fixed trace case for normal matrices differs from the non-hermitian one.
- **Almost hermitian matrices**: The model can be generalized by considering matrices of the form $\mathbf{J} = \mathbf{H}_1 + \mathbf{i}\mathbf{v}\mathbf{H}_2$ with $\mathbf{H}_1$ and $\mathbf{H}_2$ being mutually independent hermitian random matrices with i.i.d. matrix entries. The weight function in this case is given by

$$\mathbf{w}(\mathbf{J},\mathbf{t}) = \left(\frac{\mathbf{N}}{\pi\sqrt{\mathbf{1}-\tau^2}}\right)^{\mathbf{N}^2} \delta\left(\mathbf{t} - \frac{\mathbf{N}}{\mathbf{1}-\tau^2}\mathrm{Tr}\left[\mathbf{JJ}^{\dagger} - \tau\mathrm{Re}\mathbf{J}^2\right]\right),$$

where the parameter $\tau := \frac{\mathbf{1}-\mathbf{v}^2}{\mathbf{1}+\mathbf{v}^2}$, $\mathbf{0} \leq \tau \leq \mathbf{1}$ controls the degree of non-hermiticity. For $\tau = \mathbf{0}$ we have maximum non-hermiticity and are back in the case of Ginibre. On the other hand we are back to the ensemble of hermitian matrices when $\tau \to \mathbf{1}$.
In the same way as we used the Ginibre ensemble for the computation of the $\mathbf{k}$-point function in the non-hermitian fixed trace ensemble, i.e. via the Laplace transformation, we can use the results given by [4] to calculate the $\mathbf{k}$-point function for finite $\mathbf{N}$ for the generalized fixed trace case. Applying the Poisson approach we can show once more the universality in the limit of large matrices $\mathbf{N} \to \infty$.

Summary and Conclusion

- We computed that the $\mathbf{k}$-point function for the fixed trace ensemble in the case of non-hermitian as well as for normal matrices for finite $\mathbf{N}$.
- Furthermore we showed that in the limit of large matrices $\mathbf{N}$ the behavior of the fixed trace non-hermitian matrices tend into the one of normal matrices due to the fact that they turn into the Ginibre ensemble respectively.
- Moreover we computed the finite $\mathbf{N}$ $\mathbf{k}$-point function for random matrices of the generalized form and showed universality.
- : For $\mathbf{N} \to \infty$ the Ginibre ensemble is invariant under the inverse Laplace transformation, this is also true for hermitian matrices.

References

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