Chern-Simons matrix model, Coulomb systems and SYM theories

Georgios Giasemidis[†], Miguel Tierz[*]

[†]Rudolf Peierls Centre for Theoretical Physics, Oxford, UK, giasemidis@physics.ox.ac.uk

[*]Departamento de Análisis Matemático, Universidad Complutense de Madrid, tierz@mat.ucm.es

UNIVERSITY OF OXFORD

I. Introduction

We study aspects of two different supersymmetric Yang-Mills theories by interpreting the respective partition functions in terms of lower-dimensional Coulomb systems. We then use tools from statistical mechanics to study the partition functions. In both cases, we show the relevance of the Hermitian random matrix model that describes $U(N)$ Chern-Simons theory on S^3 and its relationship with a one-component Coulomb plasma on a cylinder.

II. Strong Coupling Regime of SYM on S^5

- The partition function of $\mathcal{N}=1$ SYM on S^5 in the strong 't Hooft's coupling limit, i.e. $\lambda \equiv g_{YM}^2 N/r$, studied in [1] is given by

$$Z_{SYM} \sim \int \prod_{i=1}^{N} d\phi_i \exp\left[-\frac{4\pi^3 N}{\lambda}\sum_{i=1}^{N}\phi_i^2 + \frac{9\pi}{4}\sum_{i<j}|\phi_i-\phi_j|\right] \quad (1)$$

- The free energy has the following large N behaviour

$$F_{SYM} = -\frac{27}{256}\frac{g_{YM}^2}{\pi r}N^3 \quad (2)$$

which 1) has the "mysterious" N^3 asymptotics and 2) the numerical pre-factor is *not* consistent with its gravity dual on $AdS_7 \times S^4$.

III. One-component plasma on line

- Partition function (1) is the partition function of a statistical system of 1D one-component plasma, studied by Baxter (1963).
- Solving (1) by identifying its parameters with the parameters of 1D plasma we find indeed (2) at large N.

IV. Strong coupling of Chern-Simons MM

- The Hermitian matrix model of $U(N)$ Chern-Simons theory on S^3 is

$$Z_{CS} = \frac{e^{-g_s N(N^2-1)/12}}{N!}\int \prod_{i=1}^{N}\frac{du_i}{2\pi}e^{-u_i^2/2g_s}\prod_{i<j}\left(2\sinh\frac{u_i-u_j}{2}\right)^2 \quad (3)$$

- By making a change of variables $u_i/\sqrt{2g_s} \to u_i$ and taking the strong coupling limit, i.e. $g_s \to \infty$ we find expression (1) under the identification of the parameters

$$\frac{81\lambda}{64\pi N} \equiv 2g_s \quad (4)$$

- The matrix model (3) has been solved in [2] using the Stieltjes-Wigert polynomials

$$Z_{CS} = \left(\frac{g_s}{2\pi}\right)^{N/2} e^{g_s N(N^2-1)/12}\prod_{j=1}^{N-1}\left(1-q^j\right)^{N-j} \quad (5)$$

where $q = e^{-g_s}$. Therefore the strong coupling limit of (3) corresponds to the $q \to 0$ of (5).

- In this limit only the framing contribution survives and under the identification (4) we find (2) with the same pre-factor.
- The contribution of a framing Π on Σ^3 is parametrised by an integer $s \in \mathbb{Z}$ and shifts the partition function by a phase

$$Z \to \delta(\Sigma^3,\Pi)Z, \qquad \text{where} \qquad \delta(\Sigma^3;\Pi) = e^{2\pi i s c/24} \quad (6)$$

- Can the (fractional) framing dependence modify the pre-factor in (2) to match with its gravity dual?

V. $\mathcal{N}=4$ SYM on $S^1 \times S^3$ at low temperatures

- The low temperature limit of $\mathcal{N}=4$ SYM on spatial S^3 has the partition function

$$Z_{RT<<1} = e^{-N^2 3/(16RT)}2^N\int_{\mathbb{R}^N}\prod_{j=1}^{N}dx_j\int_{[0,\pi]^N}\prod_{j=1}^{N}dy_j e^{-V(z_1,\ldots,z_N)} \quad (7)$$

where $z_j = x_j + iy_j$ and $V(z_1,\ldots,z_N) = \frac{4N\pi^2 RT}{\lambda}\sum_{j=1}^N x_j^2 - 2\sum_{j<k}\log|\sinh z_j - z_k|$

- This system has been studied in [3] using saddle point methods and the free energy at large N is

$$F_{RT<<1} = \left(\frac{3}{16}-\frac{\lambda}{12\pi^2}\right)\frac{N^2}{RT} + o(N) \quad (8)$$

VI. One-component plasma on cylinder

- Partition function (7) has the interpretation of 2D one-component plasma on cylinder with with length L and circumference W [4]

$$Z_{\text{2D jellium}} = \frac{1}{N!}\int_{\Lambda}\prod_{j=1}^{N}d^2 z_j w(z_j,\bar{z}_j)\prod_{j<k}\left|e^{2\pi z_j/W} - e^{2\pi z_k/W}\right|^2 \quad (9)$$

where $w(z,\bar{z}) = w(x) = \exp\left(-2\pi n(x^2 + x(N-1)/(nW)\right)/W^2$.

- In the thermodynamic limit, i.e. $N, L \to \infty$, (7) is proportional to the 2D jellium (9) under the identification $4N\pi^3 RT/\lambda = 2nW^2$.
- In this limit (9) is solvable and takes the form

$$Z_{\text{2D jellium}} = \left(\frac{1}{2W^2 n}\right)^{N/2}\exp\left(\pi N(N^2-1)/(6nW^2)\right), \quad (10)$$

- and the free energy of (10) implies (8) under the above identification.

VII. 2D→1D Coulomb systems

- 2D Coulomb systems in strong magnetic field behave like 1D-systems [5].
- Apply this notion to the Coulomb system described by (7) in "magnetic field" $\propto NRT/\lambda$. At large N, the radial d.o.f freeze and the system behaves like 1D Coulomb system described by the Hermitian Chern-Simons matrix model (3).
- Through the Coulomb gas interpretation we show that the $\mathcal{N}=4$ maximally SYM gauge theory on $S^3 \times S^1$ is related to the Chern-Simons topological gauge theory on S^3.

References

[1] J. Kallen, J.A. Minahan, A. Nedelin and M. Zabzine, JHEP 1210 (2012) 184, arXiv:1207.3763 [hep-th]

[2] M. Tierz, Mod. Phys. Lett. A19 (2004) 1365-1378, arXiv:hep-th/0212128

[3] S.A. Hartnoll, S.P. Kumar, Phys. Rev. D76 026005 (2007), arXiv:hep-th/0610103

[4] L. Samaj, J. Wagner, P. Kalinay, J. Stat. Phys. 117 (2004) 159-178, arXiv:cond-mat/0407346

[5] H. Azuma, S. Iso, Phys. Lett. B331 (1994) 107-113, arXiv:hep-th/9312001