Products of Independent $\beta = 4$ Ginibre Matrices
& their Spectral Correlation Functions

Jesper R. Ipsen
Department of Physics, Bielefeld University, Bielefeld, Germany

Universität Bielefeld

Ensemble & Joint Probability Density Function

We consider the product of n independent random matrices,

$$P_n = X_1 X_2 \cdots X_n.$$

Each matrix X_i is of size $2N \times 2N$, and is drawn from the induced $\beta = 4$ Ginibre ensemble. The corresponding partition function is given by

$$Z_N^{n,m} = \int |DX| \prod_{i=1}^{n} \det(X_i)^m \exp[-\operatorname{Tr} X_i^\dagger X_i],$$

where DX is the exterior product of all independent one-forms and $|DX|$ is the corresponding unorientated volume element. The partition function can be expressed as an integral over the joint probability function $\mathcal{P}_{\text{jpdf}}$ of the complex eigenvalues z_j of the product matrix P_n,

$$Z_N^{n,m} = \prod_{k=1}^{N} \int_{\mathbb{C}} d^2 z_k\, \mathcal{P}_{\text{jpdf}}(\{z_k\}) = C_N^{n,m} \prod_{k=1}^{N} \int_{\mathbb{C}} d^2 z_k\, w_n^m(z_k) \prod_{1 \le a < b \le N} |z_a - z_b|^2 |z_a^* - z_b^*|^2 \prod_{c=1}^{N} |z_c^* - z_c|^2;$$

or equivalent, the partition function may be written as a Pfaffian

$$Z_N^{n,m} = (2N)!\, C_N^{n,m} \operatorname*{Pf}_{1 \le i,j \le 2N} \left[\int_{\mathbb{C}} d^2 z\, w_n^m(z)(z^* - z)(p_{i-1}^{n,m}(z) p_{j-1}^{n,m}(z^*) - p_{i-1}^{n,m}(z^*) p_{j-1}^{n,m}(z)) \right],$$

where $p_j^{n,m}(z)$ are j-th order monic polynomials. Above $C_N^{n,m}$ is a normalization constant and the induced weight is given in terms of a Meijer G-function,

$$w_n^m(z) = \pi^{n-1} G^{n,0}_{0,n}\left({- \atop m, \ldots, m} \,\middle|\, |z|^2 \right).$$

For $n = 1$ and $n = 2$ the Meijer G-function are given as an exponential and a Bessel function, respectively:

$$w_{n=1}^m(z) = |z|^{2m} \exp[-|z|^2] \quad \text{and} \quad w_{n=2}^m(z) = 2\pi |z|^{2m} K_0(2|z|).$$

There exist a family of polynomials which are skew orthogonal with respect to the induced weight. The skew orthogonal polynomials are in monic normalization given by

$$p_{2k+1}^{n,m}(z) = z^{2k+1} \quad \text{and} \quad p_{2k}^{n,m}(z) = \sum_{i=0}^{k} \prod_{j=i}^{k-1} (2j+m)^n z^{2i}.$$

The skew orthogonal polynomials satisfy the skew orthogonal relations

$$\langle p_{2k+1}^{n,m}, p_{2\ell+1}^{n,m} \rangle_S = \langle p_{2k}^{n,m}, p_{2\ell}^{n,m} \rangle_S = 0 \quad \text{and}$$
$$\langle p_{2k+1}^{n,m}, p_{2\ell}^{n,m} \rangle_S = h_k^{n,m} \delta_{k\ell}, \quad \text{with} \quad h_k^{n,m} = 2(\pi \Gamma[2k+2+m])^n,$$

where the skew product $\langle \cdot, \cdot \rangle_S$ is the integral from inside the above given Pfaffian. The prekernel may be expressed in terms of the skew orthogonal polynomials as

$$\kappa_N^{n,m}(z, z') = \sum_{k=0}^{N-1} \frac{p_{2k+1}^{n,m}(z) p_{2k}^{n,m}(z') - p_{2k+1}^{n,m}(z') p_{2k}^{n,m}(z)}{h_k^{n,m}}.$$

All correlations functions of order $M \le N$ can be expressed in terms of the weight and the prekernel,

$$R_M^{n,m}(z_1, \ldots, z_M) = \prod_{i=1}^{M} [(z_i^* - z_i) w_n^m(z_i)] \operatorname*{Pf}_{1 \le k,\ell \le M} \begin{bmatrix} \kappa_N^{n,m}(z_k^*, z_\ell) & -\kappa_N^{n,m}(z_k^*, z_\ell^*) \\ \kappa_N^{n,m}(z_k, z_\ell) & -\kappa_N^{n,m}(z_k, z_\ell^*) \end{bmatrix}.$$

Note that there are repulsion from the real axis, which is also seen on the scatter plot to the right.

Prekernel and Correlations at Finite N

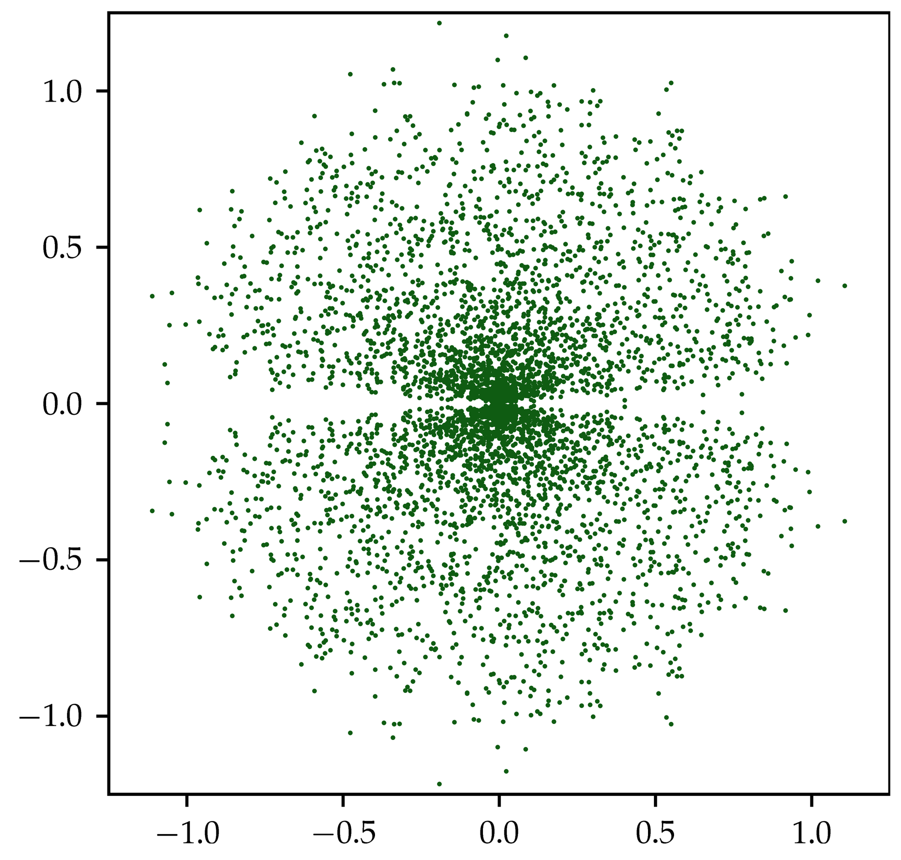
The prekernel for the product of independent $\beta = 4$ Ginibre matrices can be expressed in terms of the skew orthogonal polynomials. In the $m = 0$ case the prekernel can be expressed compactly using double factorial notation

$$\kappa_N^{n,m=0}(u,v) = \frac{1}{2\pi^n} \sum_{k=0}^{N-1} \sum_{\ell=0}^{k} \left[\frac{u^{2k+1}}{((2k+1)!!)^n} \frac{v^{2\ell}}{((2\ell)!!)^n} - \frac{v^{2k+1}}{((2k+1)!!)^n} \frac{u^{2\ell}}{((2\ell)!!)^n} \right]$$

The correlation functions are given in terms of the prekernel and the weight, e.g. the density of states and the two-point correlation function are given by

$$R_1^{n,m}(z) = (z^* - z) w_n^m(z) \kappa_N^{n,m}(z, z^*),$$
$$R_2^{n,m}(z_1, z_2) = (z_1^* - z_1)(z_2^* - z_2) w_n^m(z_1) w_n^m(z_2) \left[\kappa_N^{n,m}(z_1, z_1^*) \kappa_N^{n,m}(z_2, z_2^*) - |\kappa_N^{n,m}(z_1, z_2^*)|^2 + |\kappa_N^{n,m}(z_2, z_1^*)|^2 \right].$$

It is still an open problem to determine large-N structures for the correlation functions in the complex plane for $n \ge 3$.



Scatter Plot

Scatter plot of eigenvalues for 50 random $\beta = 4$ Ginibre matrices with $n = 3$, $m = 0$ and $N = 50$. The eigenvalues has been rescaled with a factor $(2N)^{n/2}$ such that they lie within the unit circle in the large-N limit.

Radial Behavior for Density of States at Finite N

The radial behavior of the density of states is obtained by integrating out the complex phase. Due to the fact that the complex phases $e^{i\ell\theta}$ are orthogonal with respect to any rotational invariant measure, the radial density of states becomes

$$\rho_N^{n,m}(r) = 2^n G^{n,0}_{0,n}\left({- \atop m, \ldots, m} \,\middle|\, r^2 \right) \sum_{k=0}^{N-1} \frac{r^{4k+2}}{\Gamma[2k+2+m]^n}.$$

The analytic curve on the figure below has $n = 2$, $m = 0$ and $N = 10$.

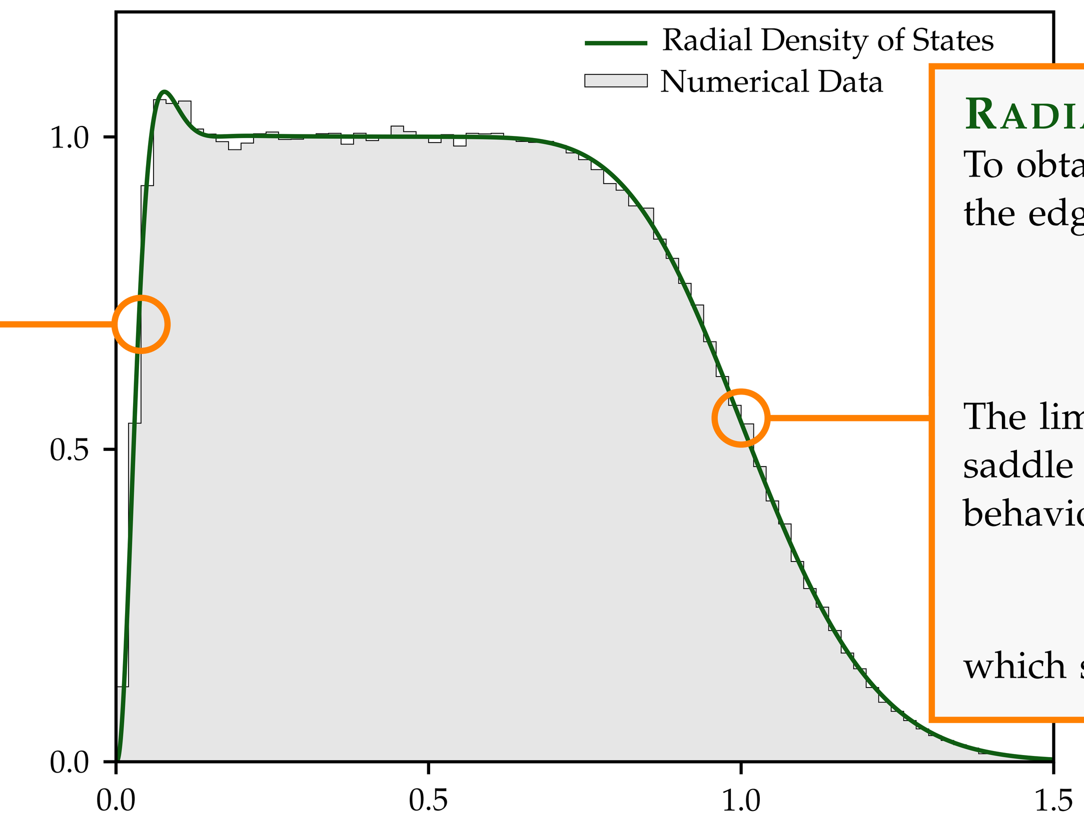
Radial Large-N Origin Limit

To obtain the large-N origin limit we let r be of order unity, while we take $N \to \infty$. The infinite sum can be written as generalized hypergeometric function,

$$\rho_{\text{origin}}^{n,m}(r) = 2^n \pi^{1-n} w_n^m(r) \frac{r^2}{\Gamma[m+2]^n} \times {}_1F_{2n}\left({1 \atop \frac{m+2}{2}, \ldots, \frac{m+2}{2}, \frac{m+3}{2}, \ldots, \frac{m+3}{2}} \,\middle|\, \frac{r^4}{2^{2n}} \right),$$

where both $(m+2)/2$ and $(m+3)/2$ appear n times in the hypergeometric function. For $m = 0$ and $n = 1, 2$ we have

$$\rho_{\text{origin}}^{n=1,m=0}(r) = 2\exp[-r^2]\sinh(r^2), \quad \text{and}$$
$$\rho_{\text{origin}}^{n=2,m=0}(r) = 4K_0(2r)[I_0(2r) - J_0(2r)].$$



Radial Large-N Edge Limit

To obtain at the large-N edge behavior we introduce the edge variable ε given by

$$r = (2N)^{n/2} \left(1 + \varepsilon \sqrt{\frac{n}{2N}} \right).$$

The limit $N \to \infty$ (and m finite) can be obtained using saddle point approximation which leads to the edge behavior

$$\rho_{\text{edge}}^{n,m}(\varepsilon) = \frac{1}{2} \operatorname{erfc}(\sqrt{2}\,\varepsilon),$$

which shows the universal behavior at the edge.

Numerical Dataset

The numerical data on the figure to the right is from an ensemble of 50 000 random $\beta = 4$ Ginibre matrices with $n = 2$, $m = 0$ and $N = 10$.

Refereces

Akemann & Burda, *Universal Microscopic Correlation Functions for products of Independent Ginibre Matrices*, J.Phys.A 45 465201 (2012).
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