IDENTITIES AND EXPONENTIAL BOUNDS FOR TRANSFER MATRICES

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Abstract: Analytic statements can be made on eigenvalues z_i and singular values σ_i of the transfer matrix T_n of a single general block tridiagonal matrix H:

1) duality identity and Thouless-like identities for $\frac{1}{n}\log|z_i|$ (exponents);

2) There are constants K, H such that

$$\sigma_i > e^{Hn+K}, \quad \sigma_{m+i} < e^{-Hn-K} \qquad i = 1...m$$

(based on: Demko S, Moss W F and Smith P W, *Decay rates for inverses of band matrices*, Math. Comp. **43** (1984) 491-499)

Block tridiagonal matrix & its transfer matrix

$$H = \begin{bmatrix} A_1 & B_1 & & C_1 \\ C_2 & \ddots & \ddots & \\ & \ddots & \ddots & B_{n-1} \\ B_n & & C_n & A_n \end{bmatrix}_{nm\times nm}$$

$$H\Psi = E\Psi \qquad \Rightarrow \qquad T_n(E)\begin{bmatrix} \psi_1 \\ \psi_0 \end{bmatrix} = \begin{bmatrix} \psi_{n+1} \\ \psi_n \end{bmatrix}.$$

$$\psi_{n+1} = \psi_1, \quad \psi_n = \psi_0$$

$$T_n(E) = \prod_{k=1}^{n} \begin{bmatrix} B_k^{-1}(E-A_k) & -B_k^{-1}C_k^\dagger \\ I_m & 0 \end{bmatrix}_{2m\times 2m}$$

The spectral duality

$$T_n(E)\begin{bmatrix}\psi_1\\ \psi_0\end{bmatrix} = z\begin{bmatrix}\psi_1\\ \psi_0\end{bmatrix} \quad \Rightarrow \quad \psi_{n+1}=z\psi_1, \quad \psi_n = z\psi_0$$

Introduce the auxiliary matrix $H(z)$ with z-b.c.:

$$H(z)=\begin{bmatrix} A_1 & B_1 & & C_1/z \\ C_2 & \ddots & \ddots & \\ & \ddots & \ddots & B_{n-1} \\ zB_n & & C_n & A_n \end{bmatrix}$$

$$\boxed{\det[T_n(E)-z] = (-z)^m \frac{\det[E-H(z)]}{\det[B_1\cdots B_n]}}$$

Thouless-like identities

Introduce the exponents of $T_n(E)$: $\xi_k =: \frac{1}{n}\log|z_k|$ $(k=1\dots 2m)$. By means of **Jensen's theorem** obtain:

$$\begin{aligned} &\frac{1}{m}\sum_{\xi_k<\xi}(\xi-\xi_k) \ -\xi \\ &= \frac{1}{mn}\int_0^{2\pi}\frac{d\varphi}{2\pi}\log\left|\det[H(e^{n\xi+i\varphi})-E]\right| \\ &- \frac{1}{mn}\sum_{j=1}^{n}\log|\det\, C_j| \end{aligned}$$

$\xi=0$ is deterministic analogous of Thouless formula for sum of exponents.

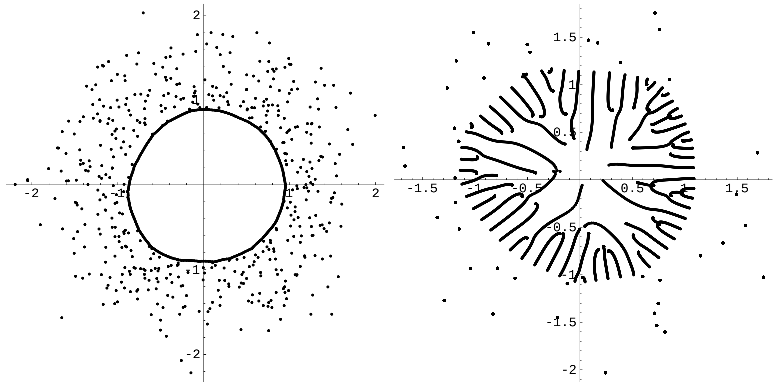

FIGURE 1. Left: eigenvalues of a tridiagonal matrix with diagonals $(e^{.5}a_k, b_k, e^{-.5}c_k)$, a_k, b_k, c_k random in [-1,1], $k = 1...800$. Right: motion of the eigenvalues in the parameter ξ; they are prgressively captured by the expanding circle $\xi(E) = \xi$ (delocalized states). Eventually all eigenvalues are on a circle.

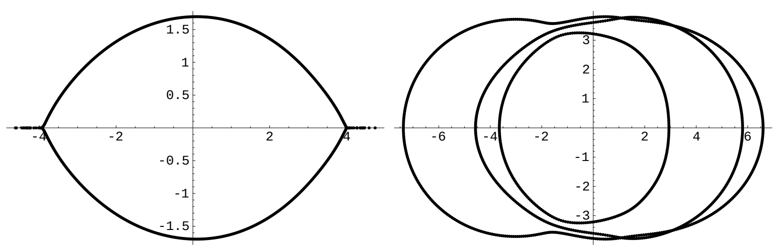

FIGURE 2. Spectral curves: tridiagonal random matrix with elements $(e, x, 1/e)$, $|x| < 3.5$ random, $n = 600$; Anderson model on strip 3x8 with b.c. $e^{n\xi+i\varphi}$ $(|xi = 1.5, w = 7)$. As $0 < \varphi < 2\pi$, the 24 eigenvalues trace curves that produce lines $\xi_k(E) = 1.5$, $k = 1, 2, 3$.

Demko-Moss-Smith

Lemma [Chebyshev]
P_k= set of real monic polynomials, $[a, b] \subset \mathbb{R}^+$

$$\inf_{p \in P_k} \left\{ \sup_{x \in [a,b]} \left| \frac{1}{x} - p(x) \right| \right\} = C\, q^{k+1},$$

$$C = \frac{(\sqrt{b} + \sqrt{a})^2}{2ab}, \qquad q = \frac{\sqrt{b} - \sqrt{a}}{\sqrt{b} + \sqrt{a}}$$

Note: Use Lemma to produce estimate for A^{-1}, where A is block tridiagonal, by noting that blocks $[i, j]$ of matrix $p_k(A)$ are null for $|i - j| > k$.

Theorem I [Demko, Moss, Smith], $A > 0$ block tridiagonal matrix,
$A^{-1}[i, j]$= any matrix element in block $(A^{-1})_{ij}$.

$$\boxed{\left|A^{-1}[i, j]\right| \leq C\, q^{|i-j|}, \qquad |i - j| \geq 1}$$

$$q = \frac{\sqrt{\mathrm{cond(A)}} - 1}{\sqrt{\mathrm{cond(A)}} + 1}, \qquad C = \frac{(\sqrt{\mathrm{cond(A)}} + 1)^2}{2\|A\|}$$

$\mathrm{cond}(A) =: \|A\|\, \|A^{-1}\| \geq 1$ (condition number)

Theorem II [Demko, Moss, Smith] A= invertible block tridiagonal matrix,
$A^{-1}[i, j]$= any matrix element in block $(A^{-1})_{ij}$.

$$\boxed{|A^{-1}[i, j]| \leq Q\, q^{\frac{1}{2}|i-j|}, \quad |i - j| > 2}$$

$Q = \frac{C}{q} \max_{ij} \|A_{i,j}\|$, C, q evaluated with $A^\dagger A$.

Theorem DMSII is used to give estimates on the singular values of the transfer matrix, whose blocks may be represented as blocks of the resolvent of H with corners removed:

transfer matrix & resolvent

$$g(E) = \begin{bmatrix} E - A_1 & -B_1 & & 0 \\ -C_2 & \ddots & \ddots & \\ & \ddots & \ddots & -B_{n-1} \\ 0 & & -C_n & E - A_n \end{bmatrix}^{-1}$$

$$T_n(E) = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}$$
$$= \begin{bmatrix} -B_n^{-1}(g_{1,n})^{-1} & -B_n^{-1}(g_{1,n})^{-1}g_{1,1}C_1 \\ g_{n,n}(g_{1,n})^{-1} & g_{n,n}(g_{1,n})^{-1}g_{1,1}C_1 - g_{n,1}C_1 \end{bmatrix}$$

Exponential bounds for singular values σ_k of T

Lemma Let t_k $k = 1...m$ be the singular values of the block T_{11} of $T_n(E)$, then:

$$t_k > \frac{1}{K} q^{-n/2}$$

Use the properties, 1) interlacing property $\sigma_k \geq t_k \geq \sigma_{m+k}$
2) T^{-1} is a transfer matrix with sing. values σ_k^{-1}
to prove:

Main theorem
If $q < 1$, the transfer matrix $T_n(E)$ has m singular values larger than $\frac{1}{K} q^{-n/2}$ and m singular values smaller than $K q^{n/2}$.