Obtaining GSE statistics without spin

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Outline

We want to investigate the eigenvalues E_n of some 'complicated' quantum Hamiltonian H, satisfying

$$H\phi_n = E_n\phi_n.$$

In the semiclassical limit, quantum systems with chaotic classical counterparts are predicted to have energy levels distributed like the ensembles of large random matrices [1].

Previously, to the best of our knowledge, all searches for GSE statistics in quantum mechanics have used particles of half-integer spin. Here, instead, we use a quantum graph with a certain discrete symmetry to find the same distribution.

Unitary symmetries

If H is symmetric under some group $\mathcal{G}$, by which we mean there exists unitary operators $U(g)$ satisfying $[U(g), H] = 0 \; \forall \; g \in \mathcal{G}$, then ϕ_n and H can be brought into a form which is block-diagonal

$$\bar{H} = R^{-1}HR = \bigoplus_{\alpha} H^{(\alpha)}.$$

(α label the irreducible representations of group $\mathcal{G}$).

For example, if $H = -d^2/dx^2 + V(x)$ where $V(-x) = V(x)$, then our eigenfunctions satisfy $\phi_n^{(\pm)}(-x) = \pm\phi_n^{(\pm)}(x)$ and so

$$H = H^{(+)} \oplus H^{(-)}.$$

Dyson's 3-fold way

Dyson says that the sub-Hamiltonian $H^{(\alpha)}$ can take only one of three forms: Complex Hermitian, real symmetric or quaternion-real [2]. The latter occur when there exists a unitary operator S satisfying $S^{-1}H^{(\alpha)}S = (H^{(\alpha)})^*$ and $S = -S^T$. This in turn implies H is comprised of 2×2 blocks of the form

$$\begin{pmatrix} z_1 & z_2 \\ -z_2^* & z_1^* \end{pmatrix} \qquad z_1, z_2 \in \mathbb{C}.$$

Random matrices of this type lead to the Gaussian Symplectic Ensemble (GSE). This happens, for example, if our full Hamiltonian is real symmetric ($H = H^*$) and $\mathcal{G}$ is given by the quaterion group.

The quaternion group and quantum graphs

The quaternion group is given by

$$Q_8 = \{\pm 1, \pm I, \pm J, \pm K : I^2 = J^2 = K^2 = IJK = -1\}$$

- All group elements can be generated by the group elements I and J.
- There exist 5 irreducible representations of this group given by $M^{(\alpha)}(I) = \pm 1$, $M^{(\alpha)}(J) = \pm 1$ $(\alpha = 1, 2, 3, 4)$ and

$$M^{(5)}(I) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad M^{(5)}(J) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

This means **any** matrix representation (meaning $M(g_1g_2) = M(g_1)M(g_2)$) of Q_8 can be decomposed into these irreducible representations, e.g.

$$R^{-1}M(I)R = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} & \\ & & & \ddots \end{pmatrix}.$$

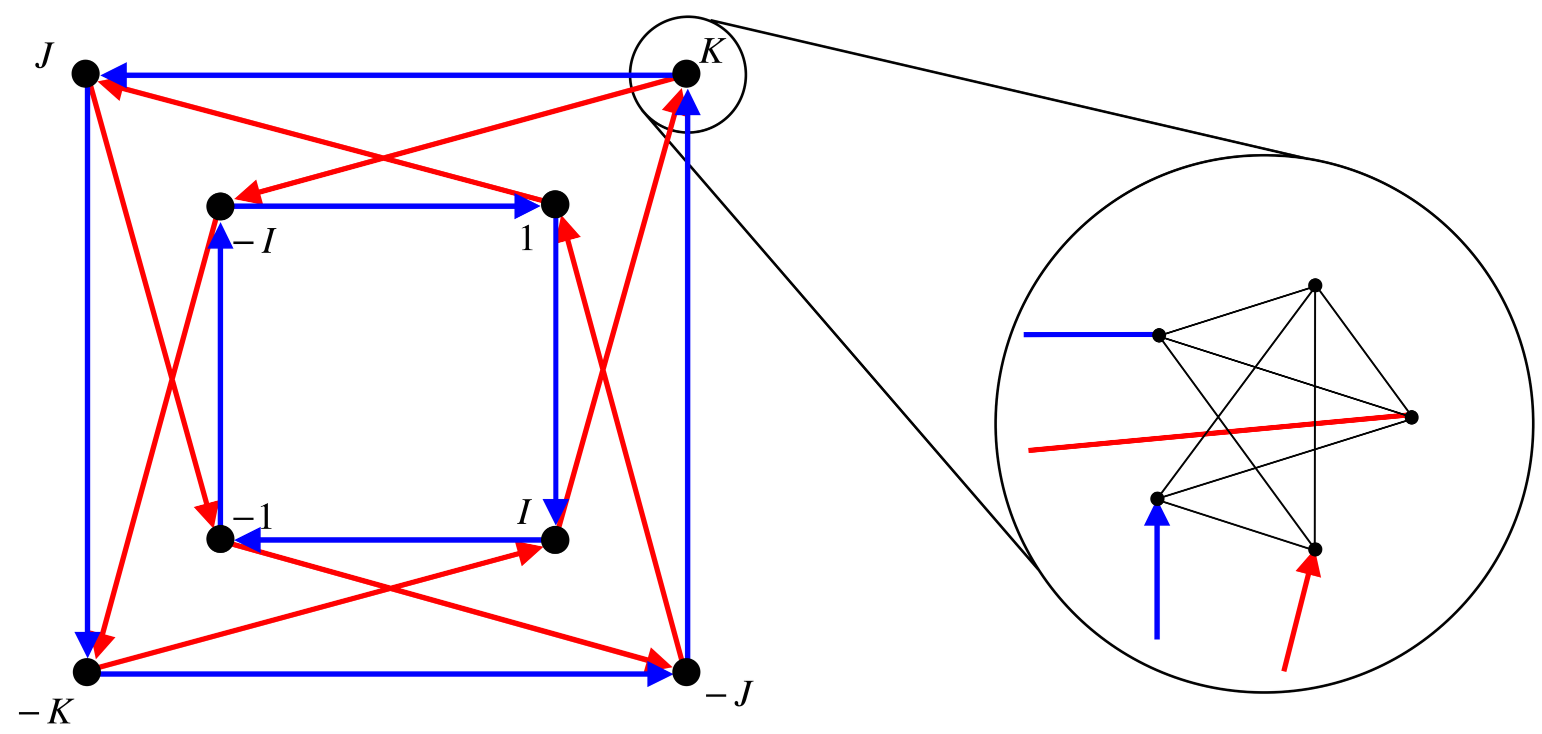
- Our first four irreducible representations are said to be **real** because all the elements are real. Our fifth is **pseudo-real** because every $M^{(\alpha)}(g)$ satisfies

$$M^{(\alpha)}(g)^* = S^{-1}M^{(\alpha)}(g)S \qquad \text{with} \qquad S = -S^T$$

- In this last case it can be shown that if $H = H^*$ then $S^{-1}H^{(\alpha)}S = (H^{(\alpha)})^*$. In other words we can pick out a **subspectrum** of H which can lead to a GSE distribution.

We can create a quantum graph with the symmetry of Q_8 by using a Cayley Graph. This is constructed as follows.

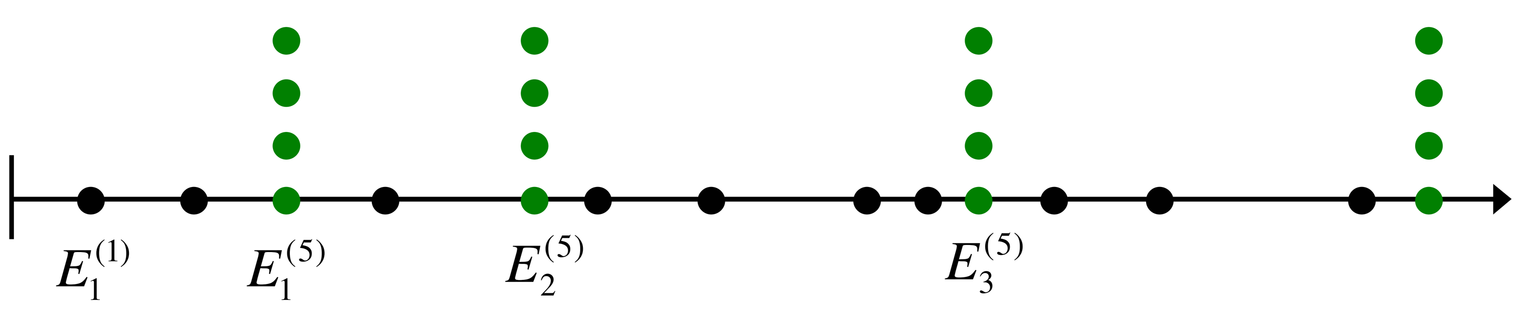
- Each vertex is given by a $g \in Q_8$.
- Each bond b is given by a generator, so $b = (g_1, g_1h)$ with $h \in \{I, J\}$.
- The graph is symmetric w.r.t. all group elements as $g \cdot b = (gg_1, gg_1h) = (g_2, g_2h)$.



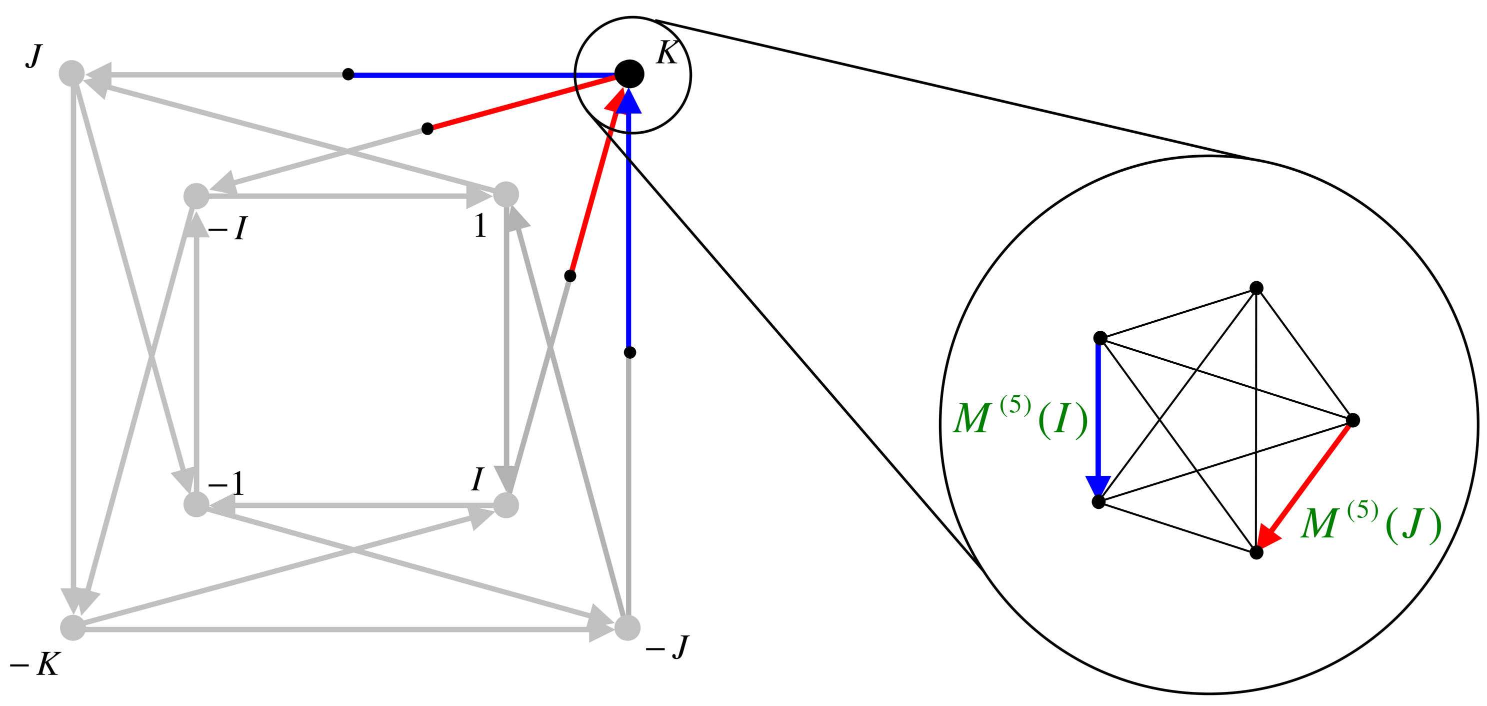
To add complexity we add more bonds at each vertex. The quantum graph is then obtained by imposing the laplacian on each bond $-d^2/dx^2\psi(x) = E\psi(x)$, subject to Neumann Vertex conditions: For all adjacent bonds

$$\psi_{b_i}(v) = \psi_{b_j}(v) \; \forall \; i, j \text{ connected to } v \qquad \sum_i \frac{d\psi_{b_i}}{dx_{b_i}}(v) = 0$$

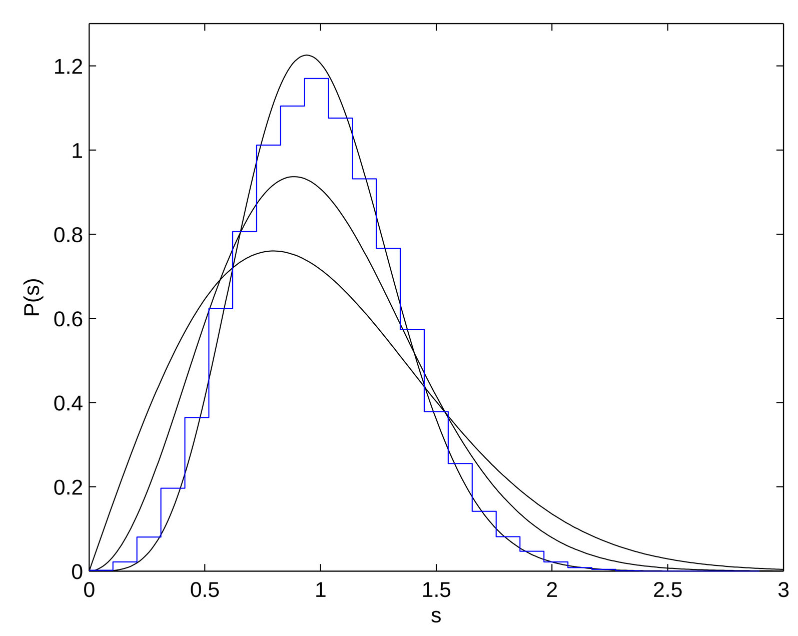
Isolating the subspectrum



Our graph has an eigenvalue spectrum containing our desired subspectrum, this is 4-fold degenerate. This can be isolated using the following procedure and in doing so emerges the relation with half-integer spin.



- All our eigenfunctions $\phi_n(x)$ can be supported using only $1/8^{\text{th}}$ of the graph - this is our '**fundamental domain**'.
- A transformation of two eigenfunctions $\phi_{1,n}^{(5)}$ and $\phi_{2,n}^{(5)}$ belonging to this subspectrum, from one fundamental domain to another are given by $M^{(5)}(g)$.
- Therefore we can isolate the fundamental domain and apply new boundary conditions - this is like splitting a system with a reflection symmetry and having Neumann/Dirichlet conditions to isolate even/odd functions.
- The result is a **quotient graph** which has only our desired spectrum (this is now 2-fold degenerate). On certain bonds there is a $SU(2)$-type phase which acts like a spin (Q_8 is a discrete subgroup of $SU(2)$) [3,4].



The nearest neighbour spacing distribution $P(s)$ of the first 20,000 energy levels. Associated to a fully connected graph with 5 vertices with 2 $M(I)$ and $M(J)$ phases (as in the graph opposite except this has 1 $M(I)$ and $M(J)$).

Conclusions and Outlook

We have created a spinless quantum system which displays a GSE distribution within a certain subspectrum. This is achieved using a discrete symmetry containing a **pseudo-real** irreducible representation.

In quantum graphs, the semiclassical limit arises when taking sequences of graphs with more bonds, although it is not immediately clear what role symmetries play within this. Their introduction is related to the GUE-GOE transition obtained when introducing a magnetic field. For instance, reducing the number of $M(I)$, $M(J)$ phases appears to agrees less with the predicted GSE distribution [5].

References

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