Self-averaging and Spectral Commutativity of Isotropic Random Matrices

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Abstract

We show that the eigenvalue distribution for the product of arbitrary number of random matrices from isotropic unitary ensemble (IUE) is independent of the order of multiplication in $N \to \infty$ limit. We present the procedure of calculating the average eigenvalue distribution for product of such matrices and employ it to numerous examples involving products of Girko-Ginibre matrices and so called "matrices with predefined eigenvalue distribution". In each case we obtain perfect agreement with Monte-Carlo simulations considering finite size effects.

Isotropic Random Matrices

Isotropic random variables:
- $z = re^{i\phi}$,
- $r \in \mathbb{R}, r \geq 0$,
- ϕ uniform on $[0, 2\pi)$.

Isotropic random matrices:
- $x = hu$,
- h - positive semi-definite Hermitian,
- u - Haar measure on $U(N)$.

In other words: For $N \times N$ Isotropic Random Matrix x one has $P(x) = P(xv)$ where $v \in U(N)$

Examples:
- Girko-Ginibre matrix
- Matrix of the form vhu where v and u are Unitary Haar measure random matrices and h is positive semi-definite hermitian random matrix.

Properties:
- Eigenvalue spectrum is rotationally symmetric on the complex plane.
- They form Isotropic Unitary Ensemble (IUE).

Order invariance of average eigenvalue spectrum

Our main result:
- Average eigenvalue distribution for the product of n matrices generated from any type of IUE is independent of the order of multiplication in the $N \to \infty$ limit.

Or in other words:
- Consider N independent IUE matrices A_i, $i = 1, 2, \ldots, N$. Defining $x = A_1 A_2 \ldots A_N$ and for any permutation π: $x_\pi = A_{\pi(1)} A_{\pi(2)} \ldots A_{\pi(N)}$ probability that the eigenvalue of x_π lies within a circle of radius r: $\text{Prob}\left(\lambda_{x_\pi} < r\right) = \text{Prob}\left(\lambda_x < r\right)$

The proof is based on Haagerup Larsen Theorem and multiplicative properties of S-transform in Free Random Variables calculus.

Haagerup-Larsen Theorem

Lets define radial cumulative density function for eigenvalues of IUE matrix x by:

$$F_x(r) = \int_0^r 2\pi |z| \rho_x(z, \bar{z})\, d|z|, \qquad (1)$$

where $\rho_x(z, \bar{z})$ is the average eigenvalue distribution. Then one has the relation between it and S-transform for h^2 from usual decomposition $x = hu$, known as Haagerup-Larsen Theorem:

$$S_{h^2}\left(F_x(r) - 1\right) = \frac{1}{r^2}. \qquad (2)$$

Additionally $S_h = S_{h'}$ if eigenvalue distribution for h are the same as of h', so one can rewrite equation (2) in terms of x only:

$$S_{x^\dagger x}\left(F_x(r) - 1\right) = \frac{1}{r^2}. \qquad (3)$$

as $h^2 = xx^\dagger$ and it has the same eigenvalues as $x^\dagger x$.

Finally, using properties of S-transforms, one has relation for $X = \prod_{i=1}^n x_i$:

$$\prod_{i=1}^n S_{x_i^\dagger x_i}\left(F_X(r) - 1\right) = \frac{1}{r^2} \qquad (4)$$

References

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- U. Haagerup and F. Larsen, Journal of Functional Analysis, 176, 331 (2000).
- D. V. Voiculescu, J. Operator Theory 18, 223 (1987).

Numerical results

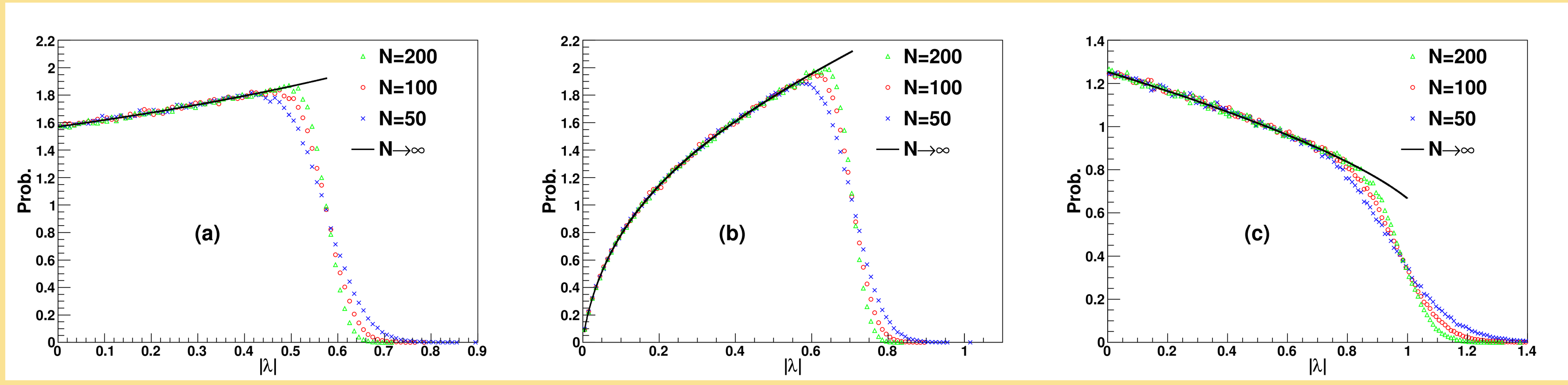

Figure: Numerical verification of theoretical predictions $p(r) = F'(r)$ (1) for the mean radial spectral density of eigenvalues for the product of independent isotropic matrices. (a) Numerical histograms for the product of independent Girko-Ginibre matrix and matrix with $\rho_h(r) = 1$ on interval (0,1) for $N = 200$ (green triangles), $N = 100$ (red circles) and $N = 50$ (blue crosses) compared with theoretical predictions for $N \to \infty$. Each histogram is made for 10^7 eigenvalues. The numerical histograms approach theoretical curve as the size of matrices is increased. (b) An analogous plot to (a) for the product of Girko-Ginibre matrix and matrix with $\rho_h(r) = 2r$ on interval (0,1). (c) An analogous plot to (a) and (b) for the product of Girko-Ginibre matrix and matrix with $\rho_h(r)$ being half-normal distribution.

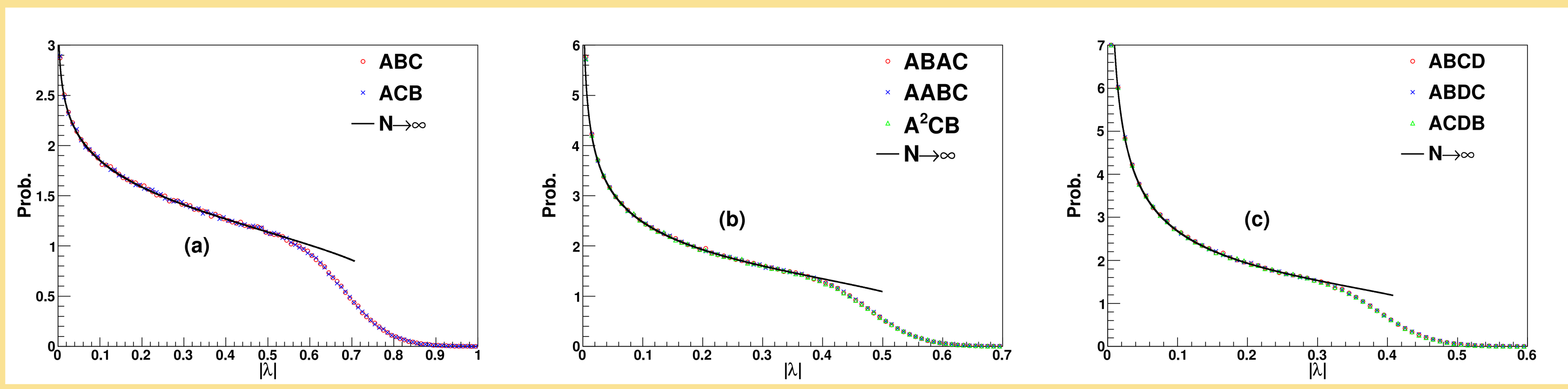

Figure: Numerical verification of theoretical predictions of the order-invariance of the mean radial spectral density, $\rho(z, \bar{z})$, of the eigenvalues for the product of independent isotropic matrices. By A we define matrix with $\rho_h(r) = 2r$ on interval (0,1), by B - Girko-Ginibre matrix, by C - matrix with $\rho_h(r)$ being half-normal distribution and by D - matrix with $\rho_h(r) = 1$ on interval (0,1). (a) Numerical histograms for the product of 3 different IUE matrices for $N = 100$, in different order compared with theoretical predictions for $N \to \infty$. ABC (red circles), ACB (blue crosses). Each histogram is made for 10^7 eigenvalues. The numerical histograms agrees with each other as well as with theoretical prediction for $N \to \infty$ limit taking into account finite size effects. (b) An analogous plot to (a) for the product of 4 IUE matrices, of which two are from same type. A_1BA_2C (red circles), A_1A_2BC (blue crosses) and A_1^2CB (green triangles). (c) An analogous plot to (a) for the product of 4 different IUE matrices. ABCD (red circles), ABDC (blue crosses) and ACDB (green triangles).

Numerical details

Having $\rho_x(r)$ or $\rho_h(r)$ ($x = hu$), one can compute the S-transform for matrix h^2, via Haagerup-Larsen Theorem or series of equations:

$$\rho_{h^2}(r) = \frac{\partial}{\partial r} \int^{\sqrt{r}} \rho_h(x)\, dx, \qquad (5)$$

$$G_{h^2}(z) = \int_{-\infty}^{\infty} \frac{\rho_{h^2}(\lambda)}{z - \lambda} d\lambda, \qquad (6)$$

$$\phi(z) = \frac{1}{z} G\left(\frac{1}{z}\right) - \frac{1}{z}, \qquad (7)$$

$$\chi(\phi(z)) = \phi(\chi(z)) = z, \qquad (8)$$

$$S_{h^2}(z) = \frac{z+1}{z} \chi_{h^2}(z). \qquad (9)$$

Having those, one can simply multiply S-transforms pointwise and use Haagerup-Larsen Theorem again to get eigenvalue distribution. Unfortunately, the procedure is almost always impossible to perform fully analytically. There are few hints for numerical evaluation:
- The cdf takes values only from $[0, 1]$ interval.
- The S-transforms have to be calculated and multiplied only on the $[-1, 0]$ interval.
- The S-transforms has to be positive on the $[-1, 0]$ interval.
- The $\chi(z)$ is always negative and on $[-1, 0]$ interval. It approaches 0 for $z \to 0^-$.
- The S-transform (and $\chi(z)$) has to be continuous and injective.

Altogether, $\chi(z)$ is a continuous, monotonically rising function. This facts allows one to specify good starting parameters to solve relevant equations numerically.