SPIN CHAINS AND RANDOM MATRIX THEORY

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MOTIVATION

The aim is to model n distinguishable spin-half particles in a ring, each only interacting with their neighbours, figure 1.

A **non-unitary invariant** random matrix ensemble with the potential to model Hamiltonians H of this system will be analysed. The average density of states and structure of **eigenstates away from the ground state** of H being calculated as $n \to \infty$.

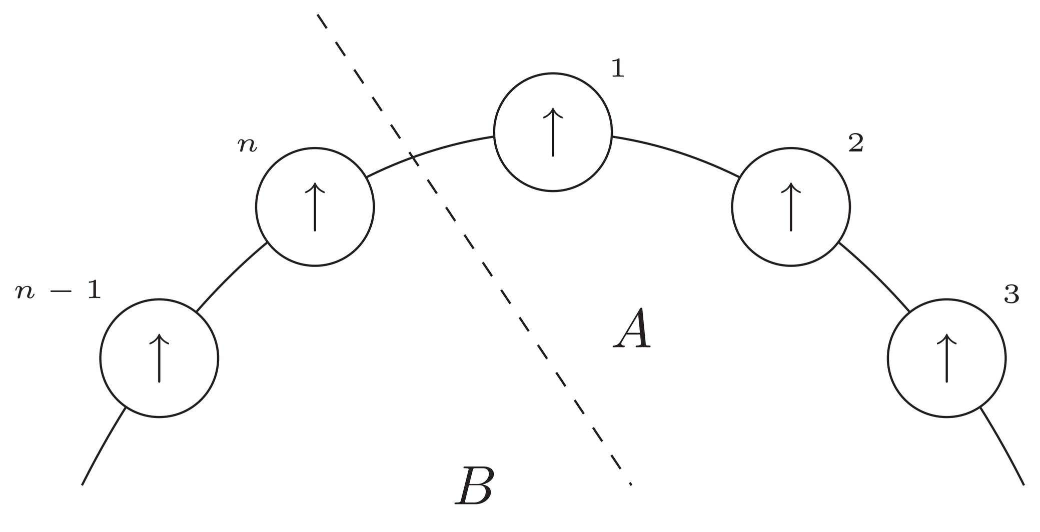

Figure 1: Spin-chain with neighbour interactions. Subsystem A has a fixed number of particles $m \leq \frac{n}{2}$.

THE MODEL

The Hilbert space $\mathcal{H}$ of the system is the n-fold tensor product of the Hilbert spaces $\mathcal{H}_i$ for the single spin-half particles, that is

$$\mathcal{H} = \bigotimes_{i=1}^{n} \mathcal{H}_i = (\mathbb{C}^2)^{\otimes n} \tag{1}$$

where subsystem A and B have Hilbert spaces

$$\mathcal{A} = \bigotimes_{i=1}^{m} \mathcal{H}_i \qquad \mathcal{B} = \bigotimes_{i=m+1}^{n} \mathcal{H}_i \tag{2}$$

The 2×2 Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

acting on the single particle spaces may be extended to act on $\mathcal{H}$ via

$$\sigma_a^{(j)} = \sigma_0^{\otimes(j-1)} \otimes \sigma_a \otimes \sigma_0^{\otimes(n-j)} \tag{3}$$

where σ_0 is the 2×2 identity, $a = 1, 2, 3$ and $j = 1, \ldots, n$.

To capture the structure of nearest neighbour interactions, define the random matrix ensemble $\mathcal{E}_n$ as the matrices

$$H = \sum_{j=1}^{n} \sum_{a,b=1}^{3} \alpha_{a,b}^{(j)} \sigma_a^{(j)} \sigma_b^{(j+1)} \tag{4}$$

where the $\alpha_{a,b}^{(j)}$ are independent $\mathcal{N}\left(0, \frac{1}{9n}\right)$ random variables and the superscript labelling is cyclic.

DENSITY MATRICES

Let $|\psi\rangle$ be a normalised vector in $\mathcal{H}$. By the **Schmidt decomposition** there exists orthonormal sets $\{|a_i\rangle\}_{i=1}^{|\mathcal{A}|} \subset \mathcal{A}$, $\{|b_j\rangle\}_{j=1}^{|\mathcal{A}|} \subset \mathcal{B}$ and non-negative scalars $\{s_k\}_{k=1}^{|\mathcal{A}|}$ such that

$$|\psi\rangle = \sum_{i=1}^{|\mathcal{A}|} s_i |a_i\rangle |b_i\rangle \tag{5}$$

Define the **density matrix** ρ associated with $|\psi\rangle$ to be $\rho = |\psi\rangle\langle\psi|$. The **reduced density matrix** $\rho_{\mathcal{B}}$ on $\mathcal{B}$ is then defined to be

$$\rho_{\mathcal{B}} = \mathrm{Tr}_{\mathcal{A}} \rho = \sum_{i,j,k=1}^{|\mathcal{A}|} s_i s_j \langle k|a_i\rangle\langle a_j|k\rangle \otimes |b_i\rangle\langle b_j| \tag{6}$$

where $\{|k\rangle\}_{k=1}^{|\mathcal{A}|}$ is any orthonormal basis of $\mathcal{A}$. In particular with $|k\rangle = |a_k\rangle$ for all k

$$\frac{1}{|\mathcal{A}|} \leq \mathrm{Tr}_{\mathcal{B}} \rho_{\mathcal{B}}^2 = \mathrm{Tr}_{\mathcal{B}} \left(\sum_{i=1}^{|\mathcal{A}|} s_i^2 |b_i\rangle\langle b_i| \right)^2 \leq 1 \tag{7}$$

with $\mathrm{Tr}_{\mathcal{B}} \rho_{\mathcal{B}}^2$ being maximal iff

$$|\psi\rangle = |\psi_{\mathcal{A}}\rangle |\psi_{\mathcal{B}}\rangle \tag{8}$$

for some $|\psi_{\mathcal{A}}\rangle \in \mathcal{A}$ and $|\psi_{\mathcal{B}}\rangle \in \mathcal{B}$ and minimal iff

$$|\psi\rangle = \sum_{i=1}^{|\mathcal{A}|} \frac{1}{\sqrt{|\mathcal{A}|}} |a_i\rangle |b_i\rangle \tag{9}$$

or equivalently if $\rho_{\mathcal{A}}$ is maximally mixed.

RESULTS

Theorem 1. *Let $\sigma(\lambda)$ be the average density of states for the ensemble $\mathcal{E}_n$, that is*

$$\sigma(\lambda) = \left\langle \frac{1}{2^n} \sum_{k=1}^{2^n} \delta(\lambda - \lambda_k) \right\rangle_{\mathcal{E}_n} \tag{10}$$

where $\{\lambda_k\}_{k=1}^{2^n}$ are the eigenvalues of H. Then for every $x \in \mathbb{R}$

$$\lim_{n \to \infty} \int_{-\infty}^{x} \sigma(\lambda)\, d\lambda = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{\lambda^2}{2}}\, d\lambda \tag{11}$$

Conjecture. *Let each $H \in \mathcal{E}_n$ have normalised eigenvectors $\{|\phi_k\rangle\}_{k=1}^{2^n}$ with corresponding eigenvalues $\{\lambda_k\}_{k=1}^{2^n}$ and denote $\rho^{(k)} = |\phi_k\rangle\langle\phi_k|$ and $\rho_{\mathcal{B}}^{(k)} = Tr_{\mathcal{A}} \rho^{(k)}$. Then*

$$\lim_{n \to \infty} \left\langle \frac{1}{2^n} \sum_{k=1}^{2^n} Tr_{\mathcal{B}} \left(\rho_{\mathcal{B}}^{(k)} \right)^2 \right\rangle_{\mathcal{E}_n} = \frac{1}{|\mathcal{A}|} \tag{12}$$

That is in the large n limit, almost all eigenvectors of almost all matrices H have a maximally mixed reduced density matrix on $\mathcal{A}$.

NUMERICS

Figure 2 shows the average frequency count of the eigenvalues of ten Hamiltonians sampled from $\mathcal{E}_{12}$. The dashed line is given by $\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}\lambda^2\right)$.

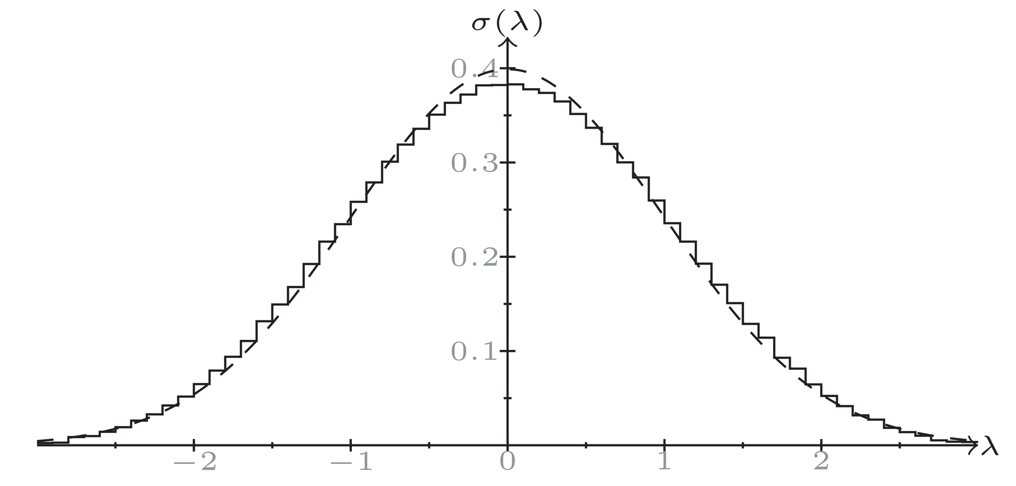

Figure 2: Numerical density of states.

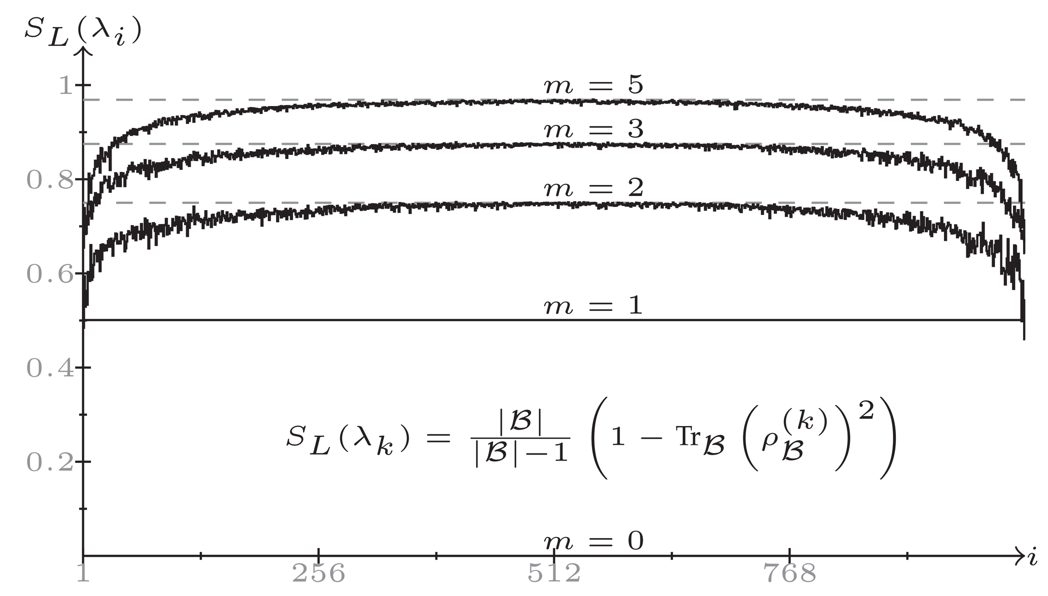

Figure 3: Numerical linear entropy.

Figure 3 shows the average eigenvector linear entropy $S_L(\lambda_k)$ of ten Hamiltonians sampled from $\mathcal{E}_{10}$ against (ordered) eigenvalue number. The dashed lines are at $S_L = 1 - 2^{-m}$.

REMARKS

Universality

The proof of Theorem 1 involves the Central Limit Theorem so the distributions of $\alpha_{a,b}^{(j)}$ need only be independent with mean zero, variance $(9n)^{-1}$ and finite fourth absolute moment.

General Interactions

The proof of Theorem 1 holds for more general particle configurations so long as the interactions are sparse enough, for example that in figure 4.

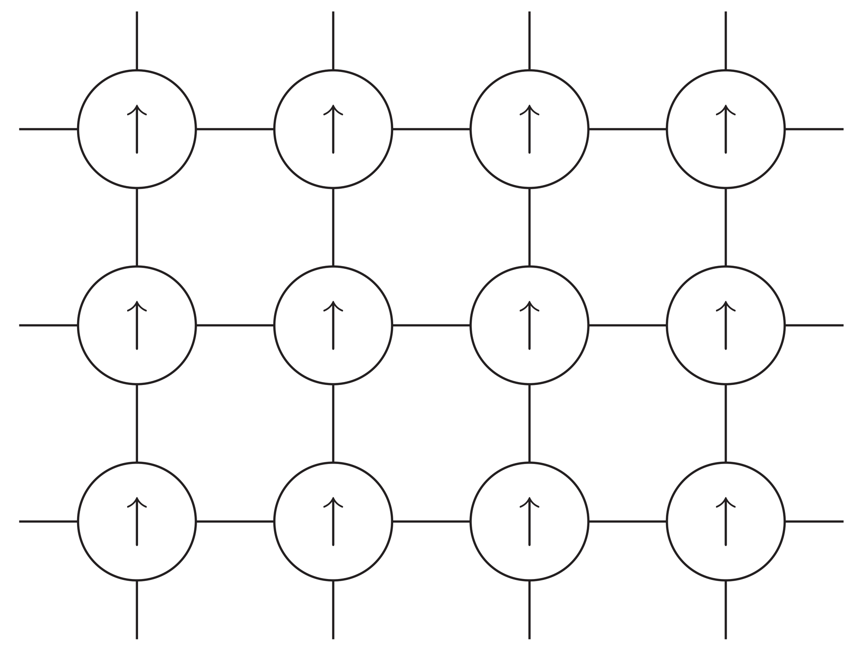

Figure 4: A spin-lattice with neighbour interactions

REFERENCES

- Hartmann, Mahler, and Hess. *Lett. Math. Phys.*, 68:103, 2004.